Small essential subgroups in fusion systems

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2. Finding all possible $\mathrm{Out}_{\mathcal{F}}(E)$
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4. Essentials hyperfocused on pearls
5. Bounding the ‘rank’ of the action on $Q/\Phi(Q)$

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You should think of a saturated fusion system $\mathcal{F}$ as a p-group P, where to each subgroup $E \leq P$ we attach a group of automorphisms $\mathrm{Aut}_P(E) \leq \mathrm{Aut}_{\mathcal{F}}(E) \leq \mathrm{Aut}(E)$. We may further assume that $\mathrm{Aut}_P(E)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E)$.

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The canonical example of a fusion system is that which a group G induces on one of its Sylow p-subgroups P, which is denoted $\mathcal{F}_P(G)$. Here we have $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}_G(E)$ for each subgroup $E \leq P$.

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One of the most important results relating to fusion systems is Alperin's fusion theorem. It tells us that to completely describe a fusion system on a p-group P, it suffices to describe $\mathrm{Aut}_{\mathcal{F}}(P)$ along with the automorphisms groups of a small number of subgroups called *essential subgroups*, or just *essentials*.

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4. $\mathrm{Out}_{\mathcal{F}}(E)$ contains a *strongly p-embedded* subgroup.

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In the second case, M and S are subject to quite strong restrictions.

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7. Compute $M := O^{p'}(H/O_{p'}(H))$ and check if M is permissible.
8. For each H with a strongly p-embedded subgroup, check if H has a preimage in $\mathrm{Out}(E)$ which intersects $O_p(\mathrm{Out}(E))$ trivially.

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If we know the behaviour of an essential isomorphic to Q with group of automorphisms $A|_Q$, then we can often deduce the behaviour of a Q-hyperfocused essential E.

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3. Pick one of minimal size whose preimage $Q \leq E$ is normalised by A and satisfies $Q \cap \Phi(E) = \Phi(Q)$.

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The presence of a pearl in a fusion system on a p-group P forces P to be maximal class. For instance, for the pearl Q_8 we must have a normaliser tower

$$Q_8 < Q_{16} < \cdots < Q_{2^{m-1}} < P \cong Q_{2^m} \text{ or } \mathrm{SD}_{2^m}.$$

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We also hope that the normaliser tower of E looks similar to the one above; i.e. each subgroup has index p in its normaliser and p of its maximal subgroups are conjugate in its normaliser.

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Proposition

Suppose that $\mathcal{F}$ is a fusion system on a 2-group P with an essential subgroup $E < P$. Suppose there exists $\mathrm{Out}_P(E) \leq A \leq \mathrm{Out}_{\mathcal{F}}(E)$ for which E is Q-hyperfocused with respect to A. Write $E = Q \times R$ if $Q \cong V_4$ or $E = Q \circ R$ otherwise, for some $R < E$. If R satisfies certain conditions then the following hold.

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While a similar result feels feasible for odd p, technical difficulties arise.

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For example, suppose $E \cong (C_p)^p$ and $N \cong C_p \wr C_p$. Then any non-trivial element of N/E acts on E with rank $p-1$, so $\mathrm{codim}(C_E(B)) = p-1$. This is less than or equal to $\dim(N/E) = 1$ if and only if $p = 2$. This reflects the fact that $(C_p)^p$ is characteristic in $C_p \wr C_p$ if and only if p is odd.

Bounding the 'rank' of the action on $Q/\Phi(Q)$

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Our lemma can be generalised to Q-hyperfocused groups E, in which case we need to check $\operatorname{codim}(C_{Q/\Phi(Q)}(B))$. However, our condition on $\Phi(E)$ must be replaced by more complicated conditions on Q and $\Phi(Q)$.

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Let $N = N_P(E)$, $d = \mathrm{rank}(E)$, $r = \mathrm{codim}(C_{E/\Phi(E)}(N/E))$ and n be the p-rank of N/E. The following result comes by considering faithful $\mathbb{F}_2$-representations of groups with a strongly embedded subgroup.

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3. *We have* $\text{rank}([y, E/\Phi(E)]) \geq n$ *for all non-trivial* $y \in N/E$.

Corollary

Suppose $E \ntrianglelefteq P$ *and* $\Phi(E)$ *is* P*-characteristic in* N. *Then* $r = n$, *and if* $n = 1$ *then* $|N : E| = 2$.

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Theorem (Baoyu Zhang, 2026+)

Let p be a prime and let $\mathcal{F}$ be a saturated fusion system on a p-group P. Suppose that $E < P$ is a 2-generated essential subgroup. Then either E is a pearl or E is a characteristic maximal subgroup of P.