

New functionalities of NumericalSgps

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What was NumericalSgps intended for?

At the beginning of the project, NumericalSgps was intended to be a package for computing with numerical semigroups (submonoids of the non-negative integers with finite complement in the non-negative integers)

- Notable elements: Apéry sets, minimal generators, Frobenius number, genus, Pseudo-Frobenius numbers, type
- Basic operations: intersection, sum, difference, sublists, elements
- Several important families: irreducible (symmetric, pseudo-symmetric), complete intersection, free, telescopic, maximal embedding dimension, Arf, saturated, among others
- Construction of numerical semigroups from others: gluing, adding a special gap, removing a minimal generator, counting ...
- Over-semigroups
- Minimal presentations, Betti elements

It has evolved based on the needs of the users and research on numerical semigroups

- New families were added: almost-symmetry and other generalizations of symmetry
- Uniquely presented numerical semigroups
- Polynomial and Hilbert series associated to numerical semigroups: cyclotomic numerical semigroups, cyclotomic exponents
- Semigroup of values of an algebraic curve
- Feng-Rao distances and numbers (divisors on semigroups; codes associated to curves)
- α -rectangular, β -rectangular, γ -rectangular, Universally free numerical semigroups
- Numerical duplications, quotients, dilatations
- Factorization invariants: factorization lengths, Delta sets, catenary degree, tame degree, ω -primality ...

Contributors

Apart from the maintainers (Manuel Delgado and Pedro A. García-Sánchez), many people have contributed to the development of the package

See the [Contributors section](#) of the [manual](#)

- Carmelo Cisto
- Benjamín Alarcón Heredia
- Juan Ignacio García-García
- Sebastian Gutsche
- Andrés Herrera-Poyatos
- Max Horn
- Helena Martin Cruz
- Nicola Maugeri
- Carlos Jesús Moreno Ávila
- Ignacio Ojeda
- Chris O’Neill
- Alessio Sammartano
- Alfredo Sánchez-R. Navarro
- Klara Stokes
- Strazzanti Francesco
- Giuseppe Zito
- Jorge Angulo Rodríguez
- Memeth Yesil
- A. Unanue Bidal

Affine semigroups and external packages

Finitely generated submonoids of \mathbb{N}^n ; they were a natural extension to consider

- Minimal generators
- Non-negative integer solutions to systems of linear Diophantine equations (and inequalities)
- Presentations
- Factorization invariants

For presentations we needed to deal with binomial ideals, and thus we decided to use Singular and 4ti2 (the user can choose)

Also factorization problems led to solving systems of linear Diophantine equations, and thus we also use Normaliz and 4ti2 for that

Good semigroups

Good semigroups emerge as a generalization of value semigroups of algebraic curves with more than one branch, and thus they are submonoids of \mathbb{N}^n that satisfy some additional properties

- Defining good semigroups
- Notable elements: small elements, conductor, genus, multiplicity, minimal good generators, Apéry sets
- Numerical good semigroup duplication
- Amalgamation of good semigroups
- Cartesian product of good semigroups
- Projections of good semigroups
- Symmetry
- Arf closure of good semigroups

Ideals

Another object related to numerical semigroups, affine semigroups and good semigroups are their ideals, which are subsets of the semigroup that are closed under addition with elements of the semigroup

- Minimal generating sets, Apéry sets, Frobenius number, genus, type
- Basic operations: intersection, sum, union, difference, sublists, elements
- Decomposition into irreducible ideals
- Blow-ups, Hilbert functions, reduction numbers, Ratliff-Rush closure, among others
- Patterns for ideals
- Properties of the graded ring associated to the semigroup ring of a numerical semigroup: Cohen-Macaulay, Gorenstein, complete intersection, Buchsbaum
- Normalized ideals (ideal class monoid of a numerical semigroup)

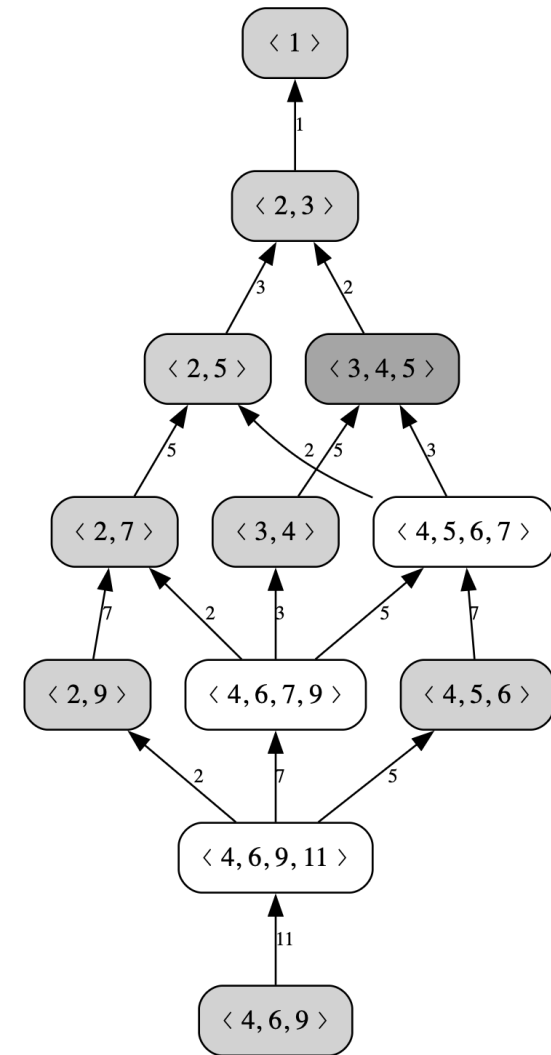
Particular functionalities for affine and good semigroup ideals

Dot functions

Presentations and factorization properties rely on graphs associated to elements of the semigroup

We wanted to have a way to visualize these graphs and also decompositions of numerical semigroups and over-semigroups

- DotBinaryRelation (with options for edges, vertices, labels, colors)
- DotTreeOfGluingsofNumericalSemigroup
- DotOverSemigroupsofNumericalSemigroup
- DotFactorizationGraph



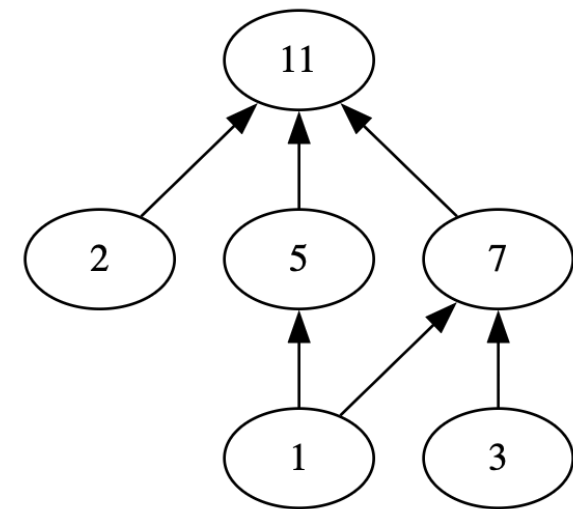
```
gap> s:=NumericalSemigroup(4,6,9);;  
gap> DotSplash(DotOverSemigroups(s));
```

Posets

A numerical seemigroup S induces an ordering on the set of integers:

$$a \leq_S b \text{ if } b - a \in S$$

- Anti-chains (improving the computation of normalized ideals)
- Maximal elements
- Minimal elements
- Upsets, downsets
- Hasse diagrams



```
gap> s:=NumericalSemigroup(4,6,9);;  
gap> g:=Gaps(s);;  
gap> p:=PosetNS(s,g);;  
gap> h:=HasseDiagram(p);;  
gap> DotSplash(DotBinaryRelation(h));
```

Numerical sets

A numerical set is a subset of the non-negative integers that contains 0 and has finite complement in the non-negative integers

For a numerical set R , the atom monoid is the set of integers n such that $n + R \subseteq R$; it is a numerical semigroup

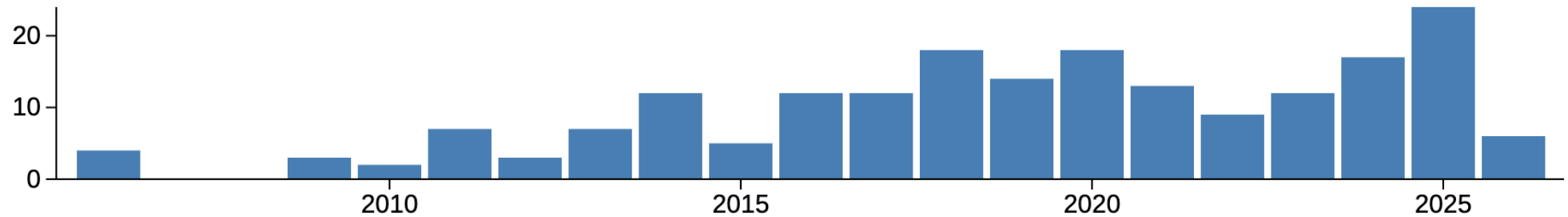
- Basic operations: intersection, union, difference, sum, sublists, elements, dual
- Notable elements: Frobenius number, genus, gaps, multiplicity, genus, conductor
- Associated numerical sets: numerical sets having a given numerical semigroup as atom monoid
- Integer partitions associated to numerical sets
- Ferrer diagrams associated to numerical sets (Young tableaux)
- Hook lengths
- Bonded sum, cojoint sum, end-to-end sum of numerical sets

```
gap> s:=NumericalSemigroup(4,6,9);;
gap> FerrersDiagram(s);
.....
...
..
.
.
.
.
[ 6, 3, 2, 1, 1, 1 ]
```

Authors and papers citing NumericalSgps

See [zbMath](#)

References by Year



- Referenced in 198 documents
- Referenced by 182 authors

More information

- [Manual](#)
- Tutorials, research, wiki on numerical semigroups, and more

numerical-semigroups.github.io

Thank you for your attention!