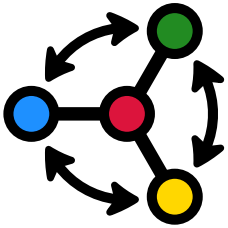

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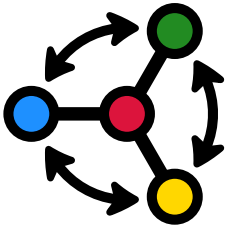
Gap Days Spring 2026

Math databases matter



Problem

Question (Hillman 2022)

What is a minimal Hirsch length $h(\Gamma)$ of a torsion-free virtually polycyclic non-solvable group Γ?

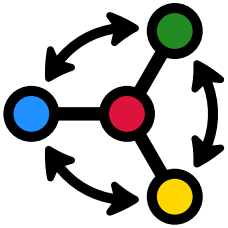
Three cases

Theorem (Hillman 2023)

1. Γ – *virtually solvable (general case):*
$$h(\Gamma) \geq 10.$$
2. Γ – *virtually nilpotent and* $h(\Gamma) \leq 14$*, then the Fitting subgroup is of nilpotency class* ≤ 3.

Theorem (L. + Szczepański 2024)

3. Γ – *virtually abelian of minimal Hirsch length:*
$$h(\Gamma) = 15.$$



Bieberbach groups

1. Problem
2. Bieberbach groups
3. Holonomy groups
4. Fast check
5. Irreducible subgroups of $\mathrm{GL}(n, \mathbb{Z})$
6. Finite perfect groups
7. Rational representations
8. Further optimization

Torsion-free virtually abelian (of rank $n \in \mathbb{N}$) group Γ

Γ — Bieberbach group

Γ is defined using

- finite holonomy group $G \subset \mathrm{GL}(n, \mathbb{Z})$
- cohomology class $\alpha \in H^2(G, \mathbb{Z}^n)$ that controls torsion-freeness

Solvability of Γ is built into G

Γ is solvable $\Leftrightarrow$ G is solvable

Non-solvable Bieberbach groups

Theorem (Hiller-Marciniak-Sah-Szczepański 1987, Plesken 1989)

1. *There exists a non-solvable Bieberbach group of dimension 15.*
2. *For holonomies $A_5, L_3(2)$ and $\mathrm{SL}_2(5)$ the dimension is at least 15.*

Idea: Go with $|G|$ as low as possible.

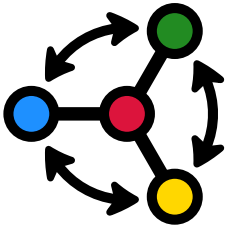
Definition

A finite non-solvable group G is minimal non-solvable (MNS) if every $H \lneq G$ is solvable.

Example: The only alternating MNS group is A_5.

Proposition

No finite MNS group is a holonomy of Bieberbach group in dimension ≤ 14.



Holonomy groups

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Holonomy of MNS Bieberbach groups

Let $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$ be the inclusion map.

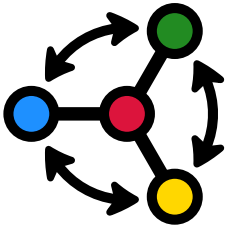
$$\varphi \sim_{\mathbb{Q}} \varphi_1 \oplus \ldots \oplus \varphi_k,$$

i.e., there exist $\varphi_i\colon G \to \mathrm{GL}(d_i,\mathbb{Z})$ such that up to conjugacy in $\mathrm{GL}(n,\mathbb{Q})$:

$$G = \left\{ \begin{bmatrix} \varphi_1(g) & & \\ & \ddots & \\ & & \varphi_k(g) \end{bmatrix} : g \in G \right\}$$

G: holonomy of a MNS Bieberbach group

- $k > 1$.
- G is MNS.
- $\varphi_i(G) \subset \mathrm{GL}(d_i,\mathbb{Z})$ is MNS group, for $1 \leq i \leq k$.



Roadmap

1. Problem
2. Bieberbach groups
3. Holonomy groups
4. Fast check
5. Irreducible subgroups of $\mathrm{GL}(n, \mathbb{Z})$
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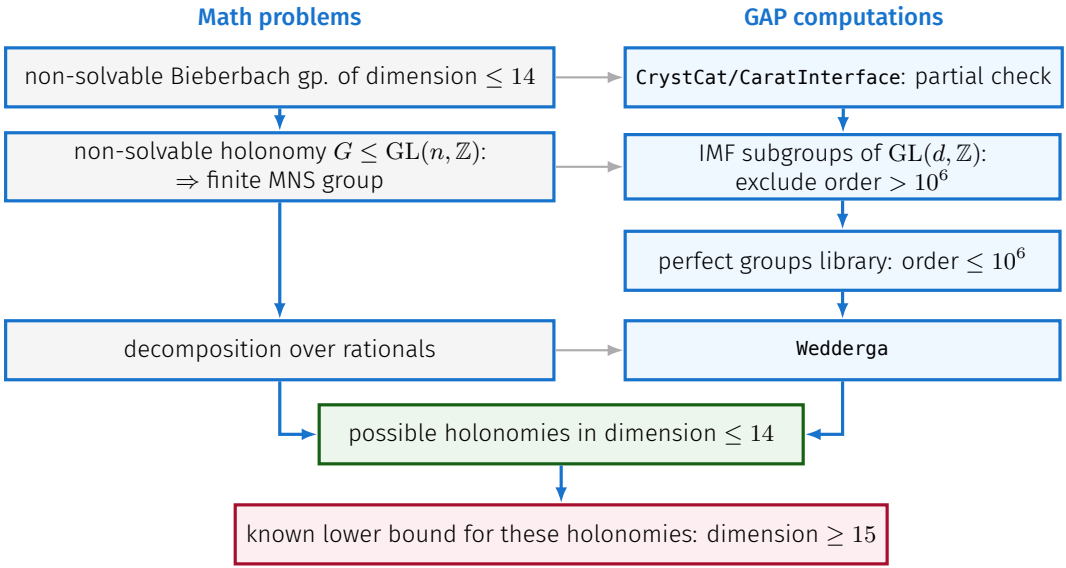
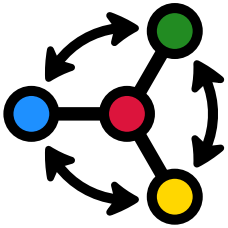
8. Further optimization

Roadmap


Math problems
GAP computations
non-solvable Bieberbach gp. of dimension ≤ 14
CrystCat/CaratInterface: partial check
non-solvable holonomy $G \leq \mathrm{GL}(n, \mathbb{Z})$:
$\Rightarrow$ finite MNS group
IMF subgroups of $\mathrm{GL}(d, \mathbb{Z})$:
exclude order $> 10^6$
perfect groups library: order $\leq 10^6$
decomposition over rationals
Wedderga
possible holonomies in dimension ≤ 14
known lower bound for these holonomies: dimension ≥ 15



Step 0: Fast check

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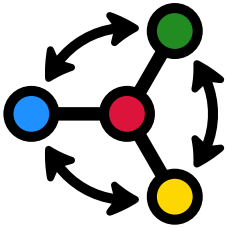
$$\varphi \sim_{\mathbb{Q}} \varphi_1 \oplus \ldots \oplus \varphi_k, \quad \varphi_i(G) \subset \mathrm{GL}(d_i, \mathbb{Z}).$$

CrystCat and CaratInterface packages:

For $n < 4$ all finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$ are solvable.

Theorem (CARAT database)

All Bieberbach groups of dimension up to 6 are solvable.



Step 1: Subgroups of $\mathrm{GL}(n, \mathbb{Z})$

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IMF subgroups of $\mathrm{GL}(n,\mathbb{Z})$

We are interested in dimensions ≤ 14:

For some $k > 1$ we have:

$$\varphi \sim_{\mathbb{Q}} \varphi_1 \oplus \ldots \oplus \varphi_k, \quad \varphi_i(G) \text{ – MNS} \quad \text{and} \quad \sum_{i=1}^{k} \deg \varphi_i(G) \leq 14$$

Idea for finding MNS groups:

Use the library of irreducible maximal finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$, for $4 \leq n \leq 10$.

Going through subgroup lattices expensive. Two approaches:

All: Check for MNS groups in the library, for $4 \leq n \leq 10$.

$> 10^6$: Check for MNS groups of order $> 10^6$ in the library, for $4 \leq n \leq 10$.

The core GAP loop

MNS subgroups calculated up to conjugacy in the whole group:

```
MaximalNonsolvableSubgroups := function(grp, min)
    return Filtered(
        MaximalSubgroupClassReps(grp),
        x -> Size(x)>=min and not IsSolvableGroup(x) );
end;
```

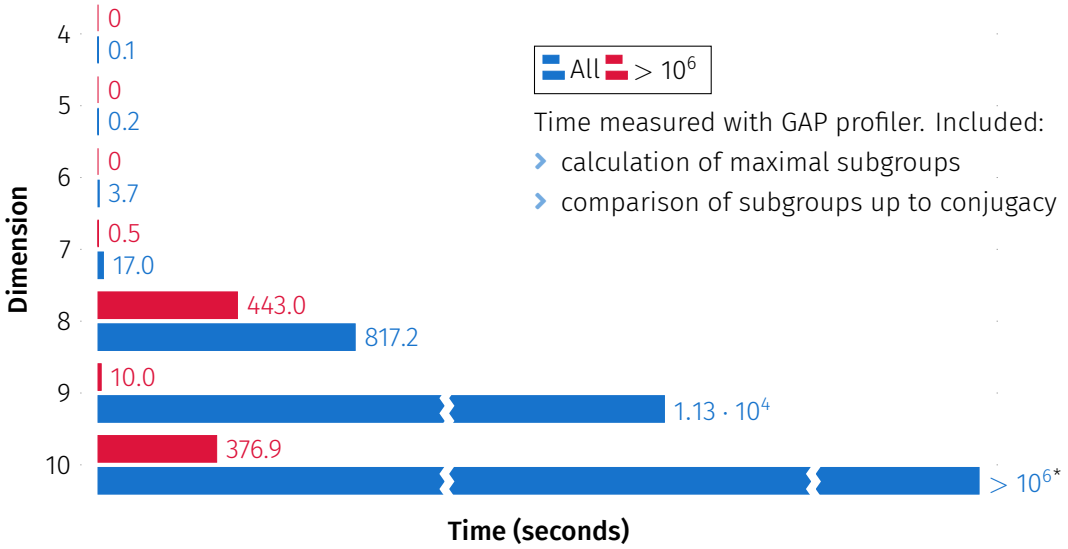
Thanks to the library, we didn't have to work with matrix groups

```
gap> G := Image( IsomorphismPermGroup( ImfMatrixGroup(10,1,1) ) );
```

All images lie in S_n

- $n \leq 270$
- median $= 42$
- average ≈ 63

Dimension 10 is the wall

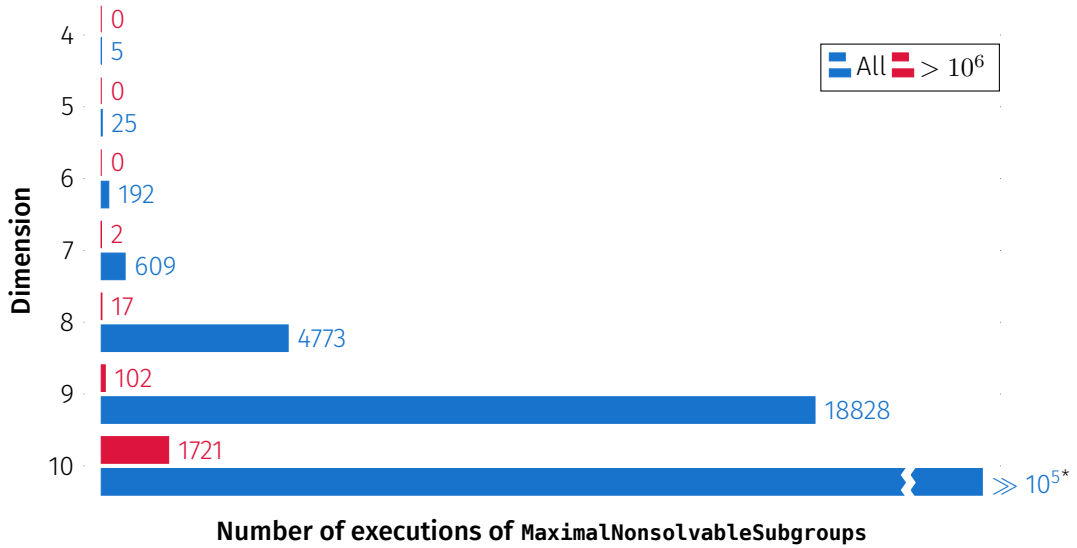


Time measured with GAP profiler. Included:

- calculation of maximal subgroups
- comparison of subgroups up to conjugacy

*with `-K 100G` switch ran out of workspace after twelve days of calculations

Count of maximal subgroups calculation

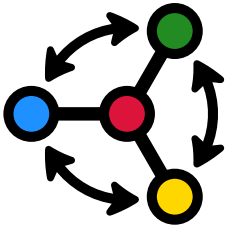


*with `-K 100G` switch ran out of workspace

Theorem

1. *Let $4 \leq n \leq 10$. There is no MNS subgroup of $\mathrm{GL}(n, \mathbb{Z})$ of order greater than 10^6.*

Question: How to handle groups of order $\leq 10^6$?



Step 2: Finite perfect groups

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Theorem

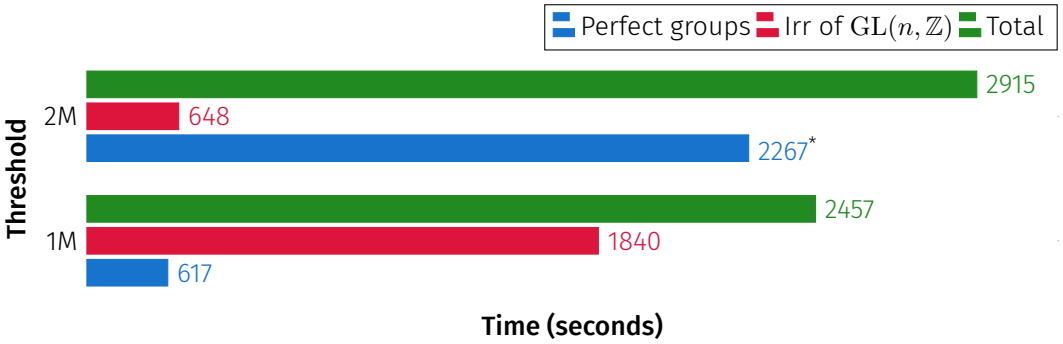
Every finite MNS group G is perfect, i.e., $G = [G, G]$.

Idea: Search for MNS groups in the library of finite perfect groups.

- Based on work of D. Holt and W. Plesken for order up to 10^6, with some gaps.
- Gaps filled and order increased up to $2 \cdot 10^6$ by A. Hulpke.

Digression

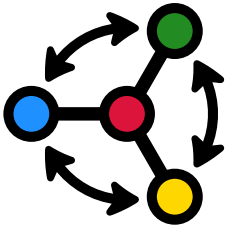
Experiment with the whole library



*Using AtlasRep for calculating maximal subgroups

Theorem

1. *Let $4 \leq n \leq 10$. There is no MNS subgroup of $\mathrm{GL}(n, \mathbb{Z})$ of order greater than 10^6.*
2. *There are 159 MNS groups of order less than or equal to 10^6 (small MNS groups).*



Step 3: Rational representations

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Rational representations of MNS groups

With GAP package Wedderga

The following small MNS groups have rational representations of dimension ≤ 14:

$$A_5, \mathrm{SL}_2(5), L_3(2), SL_2(7), L_2(8), L_3(2)N2^3.$$

With some further conditions on $\varphi\colon G \to \mathrm{GL}(n,\mathbb{Z})$ we get:

The following groups can be holonomy groups of MNS Bieberbach groups of dimension ≤ 14:

$$A_5, \mathrm{SL}_2(5), L_3(2).$$

Theorem

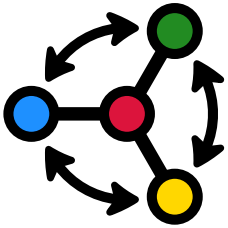
1. *Let $4 \leq n \leq 10$. There is no MNS subgroup of $\mathrm{GL}(n, \mathbb{Z})$ of order greater than 10^6.*
2. *There are 159 MNS groups of order less than or equal to 10^6 (small MNS groups).*
3. *Possible MNS holonomy groups in dimensions ≤ 14: $A_5, \mathrm{SL}_2(5)$ or $L_3(2)$.*

Back to Hiller-Marciniak-Sah-Szczepański 1987, Plesken 1989

A Bieberbach group with holonomy $A_5, \mathrm{SL}_2(5)$ or $L_3(2)$ has dimension equal at least 15.

Theorem

If Γ is a non-solvable Bieberbach group, then $h(\Gamma) \geq 15$.



Step 4: Further optimization

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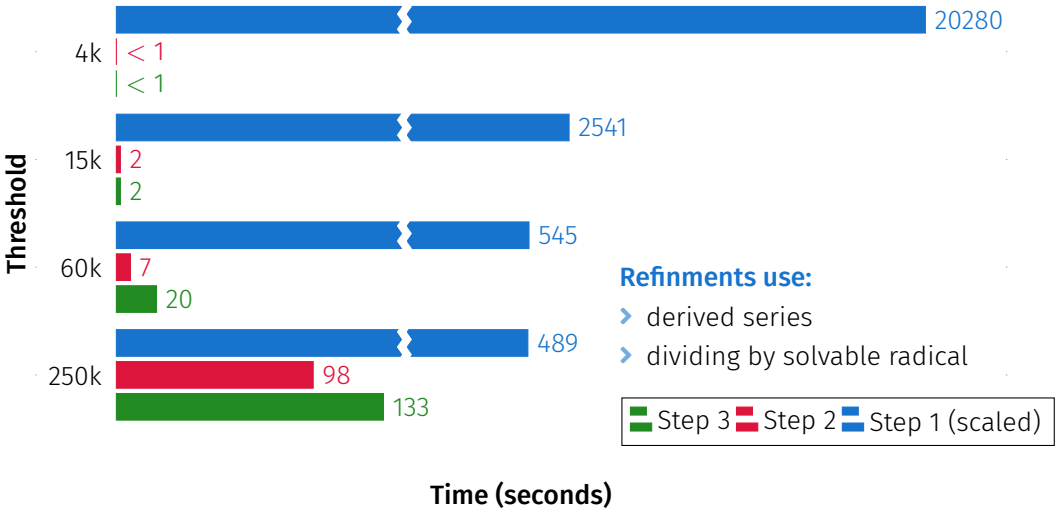
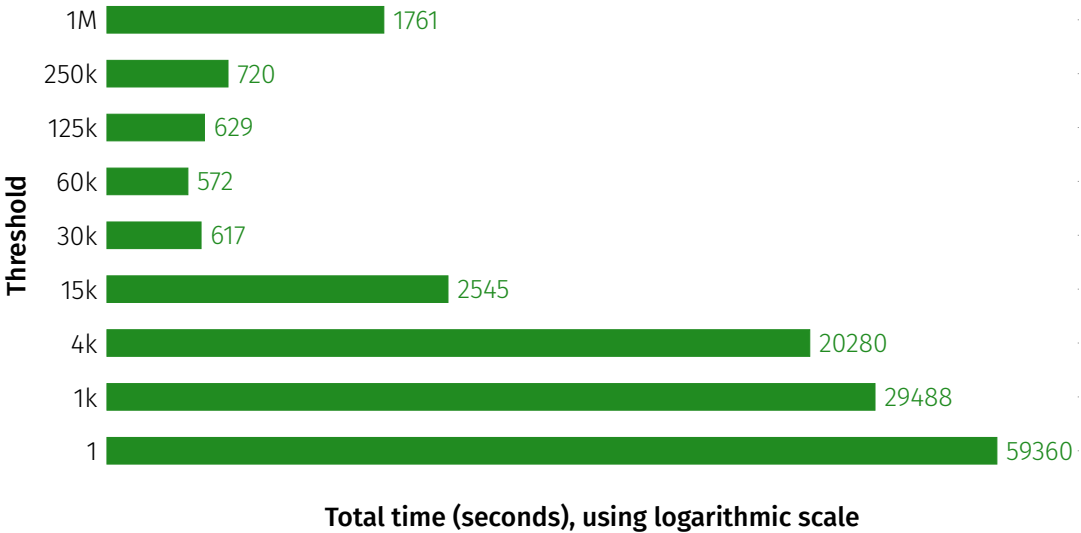
Time for fun

How much do extra mathematical filters buy us?

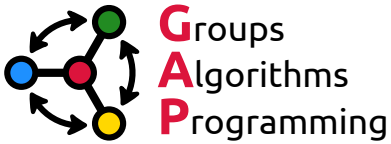


Dimension 10 is not the wall anymore!



Conclusion

Without



and

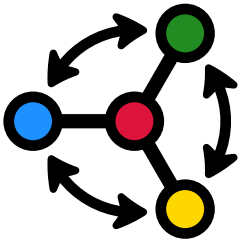
GAP libraries of:

- Irreducible Maximal Finite Integral Matrix Groups
- Finite Perfect Groups

GAP packages:

- Wedderga
- CrystCat
- CaratInterface
- AtlasRep

the result would be much harder (impossible?) to obtain.



Thank you!

https://github.com/rlutowsk/MNS-Bieberbach-groups