Fusion systems in GAP

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Table of Contents

Fusion in abelian p-groups

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Let G be a finite group and $S \in \mathrm{Syl}_p(G)$ with S abelian. Assume that $A, B \leq S$ are conjugate in G. Then A and B are conjugate in $N_G(S)$.

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So, both S and S^g are Sylow p-subgroups of $C_G(B)$.

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So, both S and S^g are Sylow p-subgroups of $C_G(B)$. By Sylow's Theorems, there exists an $h \in C_G(B)$ such that $S^{gh} = S$. Then $gh \in N_G(S)$ and $A^{gh} = B^h = B$, as desired. □

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We have actually proven something stronger. Since $h \in C_G(B)$, the conjugation map $c_h \colon B \to B$ is just 1_B. So, $c_g : A \to B$ equals c_{gh}.

Fusion in central p-subgroups

Lemma

Let G be a finite group and $S \in \mathrm{Syl}_p(G)$. Assume that $A, B \leq Z(S)$ are such that there exists an $g \in G$ with $A^g = B$. Then the conjugation map $c_g \colon A \to B$ equals $c_h \colon A \to B$ for some $h \in N_G(S)$.

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Since A and B are central subgroups of S, we have $S \leq C_G(A)$ and $S \leq C_G(B)$.

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Again, S and S^g are Sylow p-subgroups of $C_G(B)$. By Sylow's Theorems, there exists an $h \in C_G(B)$ such that $S^{gh} = S$.

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Again, S and S^g are Sylow p-subgroups of $C_G(B)$. By Sylow's Theorems, there exists an $h \in C_G(B)$ such that $S^{gh} = S$. Since $h \in C_G(B)$, we see that $c_{gh} = c_g$. Also, we have $gh \in N_G(S)$, as desired. □

Fusion

What have we found?

Let G be a finite group.

- If $S \in \mathrm{Syl}_p(G)$ is abelian, with subgroups A and B that are conjugate (*fused*) in G, then they were already conjugate in $N_G(S)$.
- In general, if $S \in \mathrm{Syl}_p(G)$, with central subgroups A and B, that are conjugate in G, then they were already conjugate in $N_G(S)$.

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General case

Let G be a finite group with $S \in \mathrm{Syl}_p(G)$, and take some conjugation map $c_g : A \to B$ for $g \in G$ and subgroups A and B of S. Does this map already lie inside $N_G(S)$? If not, can we say something about c_g?

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Fusion Systems

Definition of a fusion system

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Let G be a finite group, and S be a p-subgroup. The *fusion system* $\mathcal{F} := \mathcal{F}_S(G)$ is a category on the subgroups of S, and

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What we have found

If $S \in \mathrm{Syl}_p(G)$ is abelian, then $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$. In general, if $S \in \mathrm{Syl}_p(G)$ and $A, B \leq Z(S)$, then

$$\mathrm{Hom}_{\mathcal{F}}(A, B) = \{c_g \colon A \to B \mid g \in N_G(S), A^g = B\}.$$

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The definition of saturation does NOT require the existence of a finite group G with $S \in \mathrm{Syl}_p(G)$. As such, the natural question is – are all saturated fusion systems of this form?

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The definition of saturation does NOT require the existence of a finite group G with $S \in \mathrm{Syl}_p(G)$. As such, the natural question is – are all saturated fusion systems of this form? The answer is no! We call those not of this form as *exotic fusion systems*.

Alperin-Goldschmidt Fusion Theorem

Fusion systems on abelian groups

Let $\mathcal{F} := \mathcal{F}_S(G)$ be a fusion system, with $S \in \mathrm{Syl}_p(G)$ abelian. We have found that

$$\mathcal{F} = \mathcal{F}_S(G) = \mathcal{F}_S(N_G(S)) = \langle \mathrm{Aut}_{\mathcal{F}}(S) \rangle.$$

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Saturated fusion systems in general

Assume now that $\mathcal{F}$ is any saturated fusion system on a p-group S. We no longer have that $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S) \rangle$. However, there are some further subgroups, called *essential subgroups* such that

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E) \mid E \leq S, E \text{ essential in } \mathcal{F} \rangle.$$

This is called the **Alperin-Goldschmidt Fusion Theorem**.

Essential subgroups

Importance of essential subgroups

Let $\mathcal{E}(\mathcal{F})$ be the set of all essential subgroups of $\mathcal{F}$. We have found that

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E) \mid E \leq S, E \in \mathcal{E}(\mathcal{F}) \rangle.$$

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Properties of essential subgroups

If $E \in \mathcal{E}(\mathcal{F})$, E satisfies:

1. $C_S(E) \leq E$,
2. The largest normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E)$ is $\mathrm{Inn}(E)$, and
3. there are very few choices for $\mathrm{Aut}_{\mathcal{F}}(E)/\,\mathrm{Inn}(E)$.

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Abelian case again

Again, assume that S is abelian and let $E < S$. Then $C_S(E) = S \not\leq E$, so E does not satisfy (1). As such, $\mathcal{E}(\mathcal{F}) = \varnothing$. Thus, $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S) \rangle$ again.

Back to groups

Abelian groups

Let G be a group and $S \in \mathrm{Syl}_p(G)$ be abelian, with $A, B \leq S$. Then any conjugation $c_g \colon A \to B$ from $g \in G$ is equal to $c_h \colon A \to B$ from $h \in N_G(S)$.

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General case

Now, let G be a group, $S \in \mathrm{Syl}_p(G)$ be any group. Let $\mathcal{F} := \mathcal{F}_S(G)$, and let $\mathcal{E}(\mathcal{F})$ be the set of essential subgroups of $\mathcal{F}$.

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$$c_g = c_{x_1} \circ c_{x_2} \circ \cdots \circ c_{x_n},$$

where each $x_i \in N_G(E_i)$, with $E_i \in \mathcal{E}(\mathcal{F}) \cup \{S\}$.

Normal subgroups

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Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Then

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Let $A \leq S$. A is *normal* in $\mathcal{F}$ if $A \leq E$ for every $E \in \mathcal{E}(\mathcal{F})$, and every map $c_g \colon E \to E$ in $\mathrm{Aut}_{\mathcal{F}}(E)$ satisfies $A^g = A$.

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Normality in groups and fusion systems

Let G be a group, $S \in \mathrm{Syl}_p(G)$. Let $\mathcal{F} := \mathcal{F}_S(G)$.

- If A is normal in G, then A is normal in $\mathcal{F}$.
- If A is normal in $\mathcal{F}$, then A need not be normal in G. Examples include when S is abelian, but not normal in G.

Background Summary

Saturated fusion systems

Let G be a group and $S \in \mathrm{Syl}_p(G)$. If $\mathcal{F} := \mathcal{F}_S(G)$, then there exists a set of essential subgroups $\mathcal{E}(\mathcal{F})$ such that

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There are, however, exotic fusion systems $\mathcal{F}$ of this form, but no overgroup G exists with $S \in \mathrm{Syl}_p(G)$.
We hope that finding all 'simple' fusion systems will allow us to understand the classification of finite simple groups better. Given that the classification for groups is complete, it suffices to find exotic, simple fusion systems.
Most saturated fusion systems are NOT exotic. For example, if $\mathcal{F}$ has a normal subgroup N such that $C_S(N) \leq N$, then $\mathcal{F}$ cannot be exotic.

Table of Contents

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Use GAP to find all interesting fusion systems on a given p-group S. There are likely many fusion systems $\mathcal{F}$ on S, but very few interesting ones. This typically means that $\mathcal{F}$ has no non-trivial normal subgroups.

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We will focus on (1) now.

Table of Contents

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Proto-essential subgroups

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Previous algorithm

This is found in the MAGMA package created by Parker-Semeraro on fusion systems.

Fix a p-group S. The algorithm works as follows:

- Find all subgroups of $S/Z(S)$.
- Take their full preimage in S.
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MAJOR LIMITATION: We're considering way too many subgroups to find these proto-essentials!

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We need a better algorithm!

The new algorithm

Theorem (G. '26+)

Fix a truncated normal series of S:

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with $[Z_{i+1} : Z_i] = p$ for each i.

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with $[Z_{i+1} : Z_i] = p$ for each i. Then any $E \in \mathcal{E}(\mathcal{F})$ is conjugate to a subgroup of the form:

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2. Take their full preimage in S;
3. *Run the normal proto-essentials check* on these subgroups.

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Example

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Probably not! I haven't encountered a situation where this can occur. But proving it seems hard. We will need to establish some checks to ensure we're not in a counterexample case.

Recognising non-counterexample situation

The issue

We might have $E \in \mathcal{E}(\mathcal{F})$, but the algorithm finds some $\mathcal{F}$-conjugate E_0 such that $|N_S(E_0)| < |N_S(E)|$. E would pass the later tests, but E_0 fails them!

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These tests were good enough to work with the sporadic groups. **Worst case scenario** – we have to construct all ‘conjugates’ of E_0 to find E.

Thank You!