Cartan Matrix Implementation in Semigroups

Joseph Ruiz

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Outline

Theory of Cartan Matrices for Monoid Algebras

User Experience of Current Implementation

Future Work and Quality of Experience Improvements

Finite Dimensional Algebra over a Field

Definition

Let K be a field. A pair (A, φ) is a K-algebra if A is a ring and φ is a ring homomorphism from K to $Z(A)$, the center of A.

Example

Let K be a field and M be a finite monoid.

$$KM := \left\{ \sum_{m \in M} \lambda_m m : \lambda_m \in K \right\} \qquad \begin{aligned} \varphi : K &\to KM \\ k &\mapsto k \cdot 1_M \end{aligned}$$

Representation Theory of Algebras

A-mod

An A-module is a K-vector space V with a right A-action which satisfies the following:

$$v \cdot k1_A = kv$$

$$(v \cdot a) \cdot b = v \cdot (ab)$$

K-algebra homomorphisms

A K-representation of A is a ring homomorphism $\psi : A \to M_n(K)$ which also satisfies the following:

$$\psi(k1_A) = k \,\mathrm{IdentityMat}(n)$$

Jordan–Hölder Theorem

Let V a finite generated A-module and two composition series be given

$$\{0_V\} = U_0 \subset U_1 \subset \cdots \subset U_n = V$$

such that U_i/U_{i-1} is simple and

$$\{0_V\} = W_0 \subset W_1 \subset \cdots \subset W_m = V$$

such that W_i/W_{i-1} is simple then $n = m$ and there exists a permutation σ on $\{1, \ldots, n\}$ such that $U_i/U_{i-1} \cong W_{\sigma(i)}/W_{\sigma(i)-1}$.

Reducing to the composition factors

$$\begin{aligned} A\text{-mod} &\to \mathbb{Z}^n \\ V &\mapsto ([V : S_1], \dots, [V, S_n]) \end{aligned}$$

$[V, S_i] :=$ number of composition factors of V isomorphic to S_i

Representation Theory of Group Algebras

```
gap> G := SymmetricGroup(3);;
gap> ct := CharacterTable(G);;
gap> Irr(ct);;
gap> Display(ct);
```

Representation Theory of Group Algebras

Group representation: $\psi : G \to M_d(K)$

$$\psi \mapsto \chi_\psi := (\mathrm{Tr}(\psi(g_1^G)), \dots, \mathrm{Tr}(\psi(g_n^G)))$$

```
CT1

      2  1  1  .
      3  1  .  1

        1a 2a 3a
     2P 1a 1a 3a
     3P 1a 2a 1a

X.1      1 -1  1
X.2      2  . -1
X.3      1  1  1
```

Green's Relations

$$m \mathscr{L} n \text{ if and only if } Mm = Mn$$

$$m \mathscr{R} n \text{ if and only if } mM = nM$$

$$m \mathscr{J} n \text{ if and only if } MmM = MnM$$

Let m be in M. $\mathscr{L}_m$ will denote the $\mathscr{L}$-class of m, $\mathscr{R}_m$ will denote the $\mathscr{R}$-class of m, and $\mathscr{J}_m$ will denote the $\mathscr{J}$-class of m.

Green's Relations

$m \mathscr{H} n$ if and only if $m \mathscr{L} n$ and $m \mathscr{R} n$

$m \mathscr{D} n$ if and only if $m \mathscr{L} n$ or $m \mathscr{R} n$

Let m be in M. $\mathscr{H}_m$ will denote the $\mathscr{H}$-class of m, and $\mathscr{D}_m$ will denote the $\mathscr{D}$-class of m. For finite monoids the relations $\mathscr{J}$ and $\mathscr{D}$ coincide.

Theorem

For an idempotent e in M and a simple $K\mathscr{H}_e$-module V,

$$\mathrm{S}(e, V) := V \otimes K\mathscr{R}_e / \{v \in V \otimes K\mathscr{R}_e : vMe = 0\} \text{ is a simple } KM\text{-module.}$$

Theorem (1960 Clifford Munn and Ponizovskiĭ)

Let $J_1, .., J_m$ be the $\mathscr{J}$-classes that contain idempotents (the regular $\mathscr{J}$-classes).
Let $e_1, .., e_m$ idempotents such that e_i in J_i.
Let $V_{i,1}, \ldots, V_{i,r_i}$ be all the simple $\mathbb{F}\mathscr{H}_{e_i}$-modules where i ranges from 1 to m.
Then $\mathrm{S}(e_i, V_{i,j})$ are all the simple $\mathbb{F}M$-modules.

Character Table!

$$KM\text{-mod} \longrightarrow K\mathscr{H}_{e_1}\text{-mod} \times \cdots \times K\mathscr{H}_{e_m}\text{-mod} \longrightarrow KM\text{-mod}/\{\text{Compositon factors}\}$$

$$V \longmapsto (Ve_1, \dots, Ve_2) \longrightarrow (\chi_{Ve_1}, \dots, \chi_{Ve_m})$$

Character Table of a Monoid

```
gap> M := FullTransformationMonoid(2);;
gap> mct := MonoidCharacterTable(M);;
gap> Irr(mct);;
gap> Display(mct);
    c.1 c.2 c.3


X.1   1  -1   .
X.2   1   1   .
X.3   1   1   1
```

Character Table of a Monoid

```
gap> KnownAttributesOfObject(Irr(mct)[1]);
[ "ParentAttr", "ValuesOfMonoidClassFunction" ]
gap> ValuesOfMonoidClassFunction(Irr(mct)[1]);
[ 1, -1, 0 ]
gap>
gap> GeneralizedConjugacyClassesRepresentatives(M);
[ IdentityTransformation, Transformation( [ 2, 1 ] ), Transformation( [ 1, 1 ] ) ]
```

```
gap> KnownAttributesOfObject(Irr(ct)[1]);
[ "LENGTH", "UnderlyingCharacterTable", "ValuesOfClassFunction" ]
gap> ValuesOfClassFunction(Irr(ct)[1]);
[ 1, -1, 1 ]
gap>
gap> ConjugacyClasses(G);
[ ()^G, (1,2)^G, (1,2,3)^G ]
```

Krull-Schmidt Theorem

Let V a finite generated A-module and two direct sums be given

$$U_1 \oplus \cdots \oplus U_n \cong V$$

such that U_i is indecomposable and

$$W_1 \oplus \cdots \oplus W_m \cong V$$

such that W_i is indecomposable then $n = m$ and there exists a permutation σ on $\{1, \ldots, n\}$ such that $U_i \cong W_{\sigma(i)}$.

Cartan Matrix

Let K be a field. The Cartan matrix of a finite dimensional K-algebra A is the square matrix

$$C_A(P, S) := [P, S]$$

where P varies over $\mathrm{Pims}(A)$ and S varies over $\mathrm{Irr}(A)$. Furthermore $\mathrm{Pims}(A)$ and $\mathrm{Irr}(A)$ are ordered such that $P_i/rad(P_i) \cong S_i$.

Nicolas M. Thiéry 2012

Let M be a finite monoid and T be its character table. The bicharacter of M is defined to be the matrix

$$X_M(m, n) := \{x \in M : mxn^* = x\}$$

where n^* is an element in the conjugacy class of $(n^{\omega+1})^{-1}$ where the inverse is taken in $\mathscr{H}_{n^\omega}$. The product

$$T^{-t} X_M T^{-1}$$

is the Cartan matrix of M.

Balthazar Charles(2023) developed a method to compute X_M.

Cartan Matrix of Group Algebras

```
gap> cm := MonoidCartanMatrix(G);;
gap> Pims(cm);;
gap> Display(cm);
     X.1 X.2 X.3


P.1    1   .   .
P.2    .   1   .
P.3    .   .   1
```

Cartan Matrix of Group Algebras

```
gap> cm := MonoidCartanMatrix(G);;
gap> Pims(cm);;
gap> Display(cm);
    X.1 X.2 X.3


P.1   1   .   .
P.2   .   1   .
P.3   .   .   1
```

Semisimple by Maschke's theorem

Cartan Matrix of Monoid Algebras

```
gap> cm := MonoidCartanMatrix(M);;
gap> Pims(cm);;
gap> Display(cm);
     X.1 X.2 X.3


P.1    1   .   .
P.2    .   1   .
P.3    1   .   1
```

References

Thank you!

N. Thiéry, Cartan invariant matrices for finite monoids, in DMTCS Proceedings, 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012). DMTCS, 2012, pp. 887–898

Balthazar Charles, Computing character tables and Cartan matrices of finite monoids with fixed point counting, in Journal of Symbolic Computation (2025), v 125, https://doi.org/10.1016/j.jsc.2024.102371

Benjamin Steinberg, Representation Theory of Finite Monoids, in Springer Cham 2016, 1st edition, https://doi.org/10.1007/978-3-319-43932-7