Digraphs — Zero to Hero — Worksheet

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This worksheet contains the first set of exercises for the Minicourse "Digraphs— Zero to Hero" at GAP Days Summer 2025 `https://www.gapdays.de/gapdays2025-summer/` held at University of Primorska, Koper, Slovenia from 25-29 August 2025.

The problems are intentionally a bit terse, the idea being that you can try to figure out what to do using the manual at:

`https://digraphs.github.io/Digraphs/doc/chap0_mj.html`

or by asking us directly in the exercise session.

We thank Leonard Soicher and my colleagues in St Andrews for suggesting several of the exercises on this sheet.

1-1. (a) Construct a graph Γ which represents the railway system between the following Scottish stations: Aberdeen, Crianlarich, Dunbar, Dundee, Edinburgh Waverley, Glasgow Queen Street, Inverness, Ladybank, Mallaig, Oban, Perth, Stirling and Thurso. Take the vertex set to be the stations, and put an edge between vertices if there exists a direct rail link between the stations.
[**Hint:** I expect everyone knows the Scottish train system well enough to answer this question, for those with some GAPs in their knowledge google it!]

(b) How many edges does Γ have?

(c) Is it possible, for any pair of stations, to find a route between them using the edges of your graph?
[**Hint:** Use `DigraphIsConnected`.]

(d) If you remove Perth station from Γ and all railway lines in and out of it, does this change your answer to Problem 1-1c? What about if you remove Glasgow?
[**Hint:** Use `DigraphRemoveVertex` or `ArticulationPoints`.]

(e) Find a pair of stations that are the maximum distance from each other, in terms of the minimum number of stations between the pair.
[**Hint:** This is called the *diameter* of a graph.]

(f) What is the length of the longest cycle in Γ?
[**Hint:** Use `DigraphLongestSimpleCircuit`.]

1-2. A *triangle* in a graph is a set of 3 mutually adjacent vertices.

(a) Write a function `DigraphTriangles` that returns the list of all triangles in a graph.

(b) Find all of the connected graphs with five vertices and six edges (up to isomorphism) which contain no triangles.

1-3. There are 3 houses, H_0, H_1 and H_2, each of which needs to be connected to the electricity, gas and water supplies, E, G and W.

(a) Construct a graph Γ in GAP that represents the situation just described above.

(b) Show that Γ is bipartite.

(c) Is Γ planar?

1-4. (a) Show that a knight can visit each square of a 3×4 chess-board, though without finishing at the starting square.

(b) Repeat part (a) for a $2n - 1 \times 2n$ chess-board for some other values of n. What do you observe?

1-5. (a) Show that the Johnson graphs $J(5, 3)$ and $J(5, 2)$ are isomorphic.

(b) Is the dual of $J(5, 2)$ isomorphic to the Petersen graph?

1-6. Suppose that Γ is the dual of the Hoffman-Singleton graph.

(a) Show that every node in Γ has out-degree 42.
(b) Show that the automorphism group of Γ is the (unique) primitive permutation group of degree 50 and order 252000.
(c) Find all the cliques in Γ of size 15.
(d) Let Γ' be the graph whose vertices are the cliques from part (c), and with two such cliques joined by an edge if and only if the size of their intersection is 0 or 8.
(e) Show that Γ' is isomorphic to the Higman-Sims graph.
(f) Find:
  (i) the automorphism group of Γ';
  (ii) the clique number of Γ' and its dual;
  (iii) the chromatic number of Γ'.

1-7. Can you find a well-known graph which is not implemented in DIGRAPHS? If so, see Problem 1-10(b).

1-8. The *Erdős graph* is the graph with vertices the set of all human mathematicians and with an edge incident to "Dr A" and "Professor B" if these "Dr A" and "Professor B" have co-authored a paper. A person's *Erdős number* is the length of the shortest path in the Erdős graph from that person to Erdős.

(a) Download the Erdős graph Γ from the file:

    https://jdbm.me/downloads/erdos.s6.gz
    https://jdbm.me/downloads/erdos-labels.txt.gz

and read them into GAP:

```
D := ReadDigraphs("erdos.s6.gz")[1];
labels := EvalString(StringFile("labels.txt.gz"));;
SetDigraphVertexLabels(D, labels);
```

[Ok this isn't the entire graph, it's just the first $100,000$ vertices according to https://openalex.org. It's also limited to co-authors of papers in pure mathematics.]
(b) What are the mean and median number of co-authors among the authors in the graph?
(c) Among the authors in the graph, who are the top 3 in terms of having the most co-authors?
(d) Find as many participants in GAP days as you can in the graph, and compute their Erdős numbers. Who has the minimum Erdős number?
(e) Find all the mathematicians "Dr A" with Erdős number 2 such that the number of different paths from "Dr A" to Erdős is the maximum possible value.
(f) How many authors have the highest Erdős number among the nodes in the graph? What is the diameter of the graph?
(g) What other record breakers can you find in the graph?

1-9. (a) Download the simple connected graphs up to isomorphism with n vertices for $n = 2, \ldots, 10$ from

    https://users.cecs.anu.edu.au/~bdm/data/graphs.html

and read them into GAP;
[**Hint:** Use `ReadDigraphs` for small n and use `IteratorFromDigraphFile` for $n \geq 10$.]
(b) If a_n denotes the number of graphs with n vertices up to isomorphism and b_n denotes the number of graphs with n vertices, up to isomorphism, and trivial automorphism group, then the Erdös-Renyi Theorem [2] states that

$$\frac{b_n}{a_n} \to 1 \text{ as } n \to \infty.$$

Compute the automorphism groups of every connected graph with $n = 2, \ldots, 10$ vertices up to isomorphism, and compute the ratio b_n/a_n.
(c) For what value of n do you think the following inequality will hold:

$$\frac{b_n}{a_n} > 0.99?$$

(d) Suppose that Γ is a graph with v vertices and e edges. What is the minimum non-trivial value of $v + e$ such that the endomorphism monoid of Γ is trivial?
[**Hint:** Convince yourself that any such graph is a *core*, and use this!]

(e) Can you find a property P of connected graphs Γ such that the endomorphism monoid $\mathrm{End}(\Gamma)$ is a regular semigroup if and only if Γ satisfies property P (i.e. for every $f \in \mathrm{End}(\Gamma)$ there exists $g \in \mathrm{End}(\Gamma)$ such that $fgf = f$)? [**Anti-hint:** This is probably hard.]

1-10. Do you use any other graph software (`networkx` in python, `Graphs.jl` in Julia, magma, `Grape` in GAP)?

(a) If so, what was the last thing you used it for?

(b) Can you replicate this using `Digraphs` in GAP? If not, file an issue at:

`https://github.com/digraphs/Digraphs/issues`

(c) How easy was it replicate?

(d) How does `Digraphs` performance compare to that in the other software you used?

1-11. Pick an issue at:

`https://github.com/digraphs/Digraphs/issues`

discuss possible solutions with me or Joe, write some code to resolve the issue, and open a pull request at:

`https://github.com/digraphs/Digraphs/pulls`

1-12. Let G be the unique primitive group of degree 275 and order 1796256000.

(a) Find the stabilizer H of 1 in G.

(b) Compute the orbits of H on $\{1, \ldots, 275\}$.

(c) Let y be the minimum value in the only orbit of H of size 112 and let Γ be the graph returned by `EdgeOrbitsDigraph` with argument G and `[[1, y]]`. This graph is called the *McLaughlin graph*.

(d) Find the degree of the vertices of Γ.

(e) Find the clique number of Γ and its dual.

(f) Find a maximal clique in Γ.

(g) Show that there is 22-colouring of Γ.

(h) Form the subgraph Γ_1 of Γ consisting of all those vertices at distance 1 from vertex 1, and show that Γ_1 has an 8-colouring.

(i) Form the subgraph Γ_2 of Γ consisting of all those vertices at distance 2 from vertex 1 and show that Γ_2 has a 10-colouring.

(j) Deduce that the chromatic number of Γ is at most 18.

1-13. Download and read into GAP the digraph Γ in the file:

`https://jdbm.me/downloads/degrey.s6.gz`

This is Aubrey de Grey's example showing that the chromatic number of the plane is at least 5 from [1].

(a) Find the clique number of Γ is 3;

(b) Show that Γ has a 5 colouring, and deduce that the chromatic number of Γ is 3, 4, or 5.

(c) How many embeddings of the Moser spindle into Γ are there? Deduce that the chromatic number of Γ is either 4 or 5.

1-14. (a) Write a function `BooleanRowspace` that generates the row space of a boolean matrix. The input to the function should be a list of lists of `true` and `false`, and the output should have the same type (maybe a different number of rows) and consist of all the rows that you can obtain by taking an arbitrary union of rows in the input. The empty union is assumed to be the space containing the row vector consisting solely of `false`.

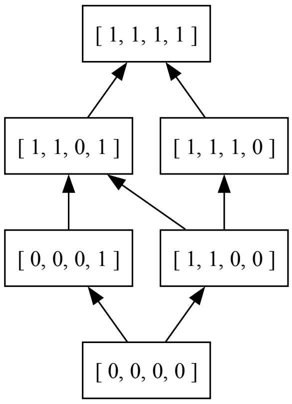
For example, the rowspace of the boolean matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

is the set of rows of

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

(b) Modify your function from part (a) to output the digraph whose vertices are the elements $\{r_1, \ldots, r_k\}$ of the row space and where there is an edge from r_i to r_j if r_i is contained in r_j in the sense of `IsSubsetBlist`. For example, the digraph corresponding to part (a) is:



(c) Download the boolean matrices from:

https://jdbm.me/koper

and read them into GAP (using `IO_CompressedFile` and `IO_Unpickle`).

(d) Verify that if A and B are any two of the matrices from (c), then the rowspace of A does not embed into the rowspace of B.

References

[1] A. D. N. J. de Grey, "The chromatic number of the plane is at least 5," *Geombinatorics* 28 (2018), 18–31.

[2] P. Erdős and A. Rényi, Asymmetric graphs *Acta Math. Hungar.* 14 (1963), 295–315.