Lazard and PORC

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Higman's PORC comjecture

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Classifying of p-groups by order

Aims

- **(Strong)** Determine up to isomorphism a complete and irredundant list of groups of order p^n.

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Classifying of p-groups by order

Aims

- **(Strong)** Determine up to isomorphism a complete and irredundant list of groups of order p^n.
- **(Weaker)** Determine the number $f(n,p)$ of isomorphism types of groups of order p^n.
- **(Variation)** Investigate $f(n,p)$ as a function in n or as a function in p.

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- there exist $s, m \in \mathbb{N}$ and polynomials $f_0, \ldots, f_{m-1}$ so that
- $f(p) = f_i(p)$ for all $p > s$ prime with $p \equiv i \bmod m$.

Example

Example for a PORC function

$$p \mapsto 2p + 61 + (p-1, 4) + 2(p-1, 3)$$

choose $s = 2$, use $m = 12$

- $f_1(p) = 2p + 61 + 10$
- $f_3(p) = 2p + 61 + 4$
- $f_5(p) = 2p + 61 + 6$
- $f_7(p) = 2p + 61 + 8$
- $f_9(p) = 2p + 61 + 6$
- $f_{11}(p) = 2p + 61 + 4$

Higman (1960)

PORC Conjecture (Higman 1960)

The function $f(n, p)$ for fixed n as a function in p is PORC.

Groups of order p^n, $p > 5$

	Number	Comment
p^1	1	
p^2	2	
p^3	5	
p^4	15	
p^5	$2p + 61 + (p-1,4) + 2(p-1,3)$	Bagnera 1898
p^6	$3p^2 + 39p + 344 + 24(p-1,3) + 11(p-1,4) + 2(p-1,5)$	Newman, O'Brien, Vaughan-Lee 2004
p^7	$3p^5 + \dots$	O'Brien, Vaughan-Lee 2005

The Lazard correspondence

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Let $p > c$:

- The Lazard correspondence associates to each group G of order p^n and p-class c a Lie ring L of order p^n and p-class c and vice versa.

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- The Lazard correspondence associates to each group G of order p^n and p-class c a Lie ring L of order p^n and p-class c and vice versa.
- The translation is done by the Baker-Campell-Hausdorff formula and its inverse.
- The correspondence preserves isomorphism.

Advantage

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Lie p-rings are slighly easier to study.

Classification

O'Brien and Vaughan-Lee (2005)

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- Determined all nilpotent Lie rings of order at most p^7 for all primes $p > 5$ up to isomorphism.
- Classification is mainly by hand and checked by computer.

Result 1

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The PORC conjecture holds for $n \leq 7$.

Result 2

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The classification is available in GAP in the LiePRing package.

The LiePRing Package of GAP

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Classification

Database of LiePRings

```
gap> List([1..7], x -> Length(LiePRingsByLibrary(x)));
[ 1, 2, 5, 15, 75, 542, 4773 ]

gap> L := LiePRingsByLibrary(5)[55];
<LiePRing of dimension 5 over prime p with parameters [ x ]>
gap> NumberOfLiePRingsInFamily(L);
1/2*p-1/2
```

Classification

Database of LiePRings

```
gap> List(LiePRingsByLibrary(5), NumberOfLiePRingsInFamily);
[ 1, 1, 1, 1, 1, 1, 1/2*p+1/2, 1, 1, 1, 1/2*p-1/2, 1, 1, 1,
  1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1/2*(p-1,3)-1/2,
  1/2*(p-1,3)-1/2, 1, 1, 1, 1/2*(p-1,4)-1, 1/2*(p-1,4)-1,
  1, 1/2*(p-1,3)-1/2, 1/2*(p-1,3)-1/2, 1, 1, 1, 1, 1, 1,
  1, 1, 1, 1, 1, 1, 1, 1, 1/2*p+1/2, 1, 1, 1/2*p-1/2, 1,
  1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ]
gap> Sum(last);
2*p+2*(p-1,3)+(p-1,4)+61
```

Classification

Database of LiePRings

```
gap> L := LL[70];
<LiePRing of dimension 5 over prime p>
gap> ViewPCPresentation(L);
[l2,l1] = l5

gap> K := SpecialisePrimeOfLiePRing(L, 5);
<LiePRing of dimension 5 over prime 5>
gap> PGroupByLiePRing(K);
<pc group of size 3125 with 5 generators>

gap> K := SpecialisePrimeOfLiePRing(L, 37);
<LiePRing of dimension 5 over prime 37>
gap> PGroupByLiePRing(K);
<pc group of size 69343957 with 5 generators>
```

Classification

Database of LiePRings

```
gap> L := LL[55];
<LiePRing of dimension 5 over prime p with parameters [ x ]>
gap> ViewPCPresentation(L);
p*l2 = x*l5
p*l3 = l4 + l5
[l2,l1] = l4
[l3,l1] = l5
gap> LiePRingsInFamily(L, 5);
[ <LiePRing of dimension 5 over prime 5>,
  <LiePRing of dimension 5 over prime 5> ]
gap> LiePRingsInFamily(L, 11);
[ <LiePRing of dimension 5 over prime 11>,
  <LiePRing of dimension 5 over prime 11>,
  <LiePRing of dimension 5 over prime 11>,
  <LiePRing of dimension 5 over prime 11>,
  <LiePRing of dimension 5 over prime 11> ]
```

Classification

Database of LiePRings

```
gap> LL := LiePRingsByLibrary(7);;
gap> L := LL[122];
<LiePRing of dim 7 over prime p with parameters [x,y,z]>
gap> L!.LibraryConditions;
[ "x ne 0, [x,y,z]~[ax,a^2y,az] if a^4=1", "" ]
gap> Length(LiePRingsInFamily(L, 11));
605
gap> Length(LiePRingsInFamily(L, 37));
12321
gap> NumberOfLiePRingsInFamily(L);
-1/8*p^3*(p-1,4)+3/4*p^3+1/8*p^2*(p-1,4)-3/4*p^2
```

Classification

Database of LiePRings

```
gap> Filtered(LL, x -> Length(ParametersOfLiePRing(x))>8);
[ <LiePRing of dimension 7 over prime p with parameters
    [ x, y, z, t, j, k, m, n, r, s, u, v ]>,
  <LiePRing of dimension 7 over prime p with parameters
    [ x, y, z, t, j, k, m, n, r, s, u, v ]> ]

gap> NumberOfLiePRingsInFamily(last[1]);
p^5+p^4+4*p^3+6*p^2
     +p*(p-1,3)+15*p+3/2*(p-1,3)+1/2*(p+1,3)+14
```

Algorithms

Computations with LiePRings

```
gap> L;
<LiePRing of dim 7 over prime p with parameters [x,y,z]>
gap> LiePLowerCentralSeries(L);
[ <LiePRing of dim 7 over prime p with parameters [x,y,z]>,
  <LiePRing of dim 5 over prime p with parameters [x,y,z]>,
  <LiePRing of dim 4 over prime p with parameters [x,y,z]>,
  <LiePRing of dim 3 over prime p with parameters [x,y,z]>,
  <LiePRing of dim 2 over prime p with parameters [x,y,z]>,
  <LiePRing of dim 0 over prime p with parameters [x,y,z]> ]
gap> List(last, BasisOfLiePRing);
[ [ l1, l2, l3, l4, l5, l6, l7 ],
  [ l3, l4, l5, l6, l7 ],
  [ l4, l5, l6, l7 ],
  [ l5, l6, l7 ],
  [ l6, l7 ],
  [  ] ]
```

What next?

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Open Problems

My favourite open problems:

- Invent a generic Lie p-ring generation algorithm

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My favourite open problems:

- Invent a generic Lie p-ring generation algorithm
- Takes as input an generic LiePRing
- Determines parametrised presentations for descendants

How to do that?

What are the problems

- Use the ideas of p-group generation:

How to do that?

What are the problems

- Use the ideas of p-group generation:
- Step (1): Compute the p-cover, the p-multiplicator, the p-nucleus

How to do that?

What are the problems

- Use the ideas of p-group generation:
- Step (1): Compute the p-cover, the p-multiplicator, the p-nucleus
- Step (2): Compute orbits and stabilizer of the automorphism group action