GAP Package - WPE

Wreath Product Elements

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Overview

WPE

provides efficient methods for working with

Wreath **P**roduct **E**lements.

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Wreath **P**roduct **E**lements.

Goals

- Intuitive research with wreath products
- Efficient computations with wreath products

Wreath Product

Definition : Primitive Permutation Group

Let $G \leq \mathrm{Sym}(n)$ be transitive.

- A subset $B \subseteq \{1, \ldots, n\}$ is a **block**, if for all $g \in G$ we have
$$B \cap B^g = \emptyset \quad \text{or} \quad B = B^g \,.$$
- G is **imprimitive**, if there exists a block B with $1 < |B| < n$.
- Otherwise, G is **primitive**.

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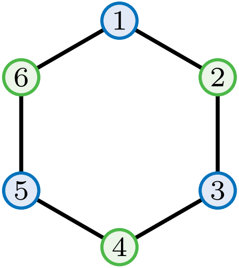
Lemma

Let $G \leq \mathrm{Sym}(n)$ be transitive. If B is a block, the orbit of B,
$$\{B^g : g \in G\},$$
is a partition (or block system) of $\{1, \ldots, n\}$.

$$\mathrm{Dih}(12) = \langle\ (1{,}2{,}3{,}4{,}5{,}6),\ (2{,}6)(3{,}5)\ \rangle$$



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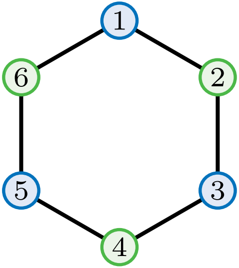

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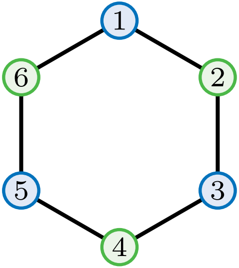

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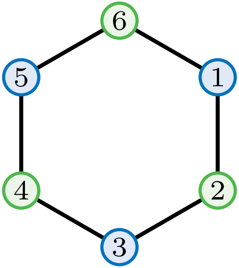

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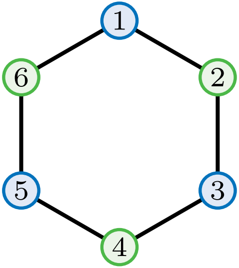

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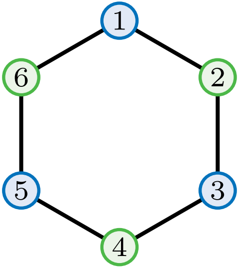

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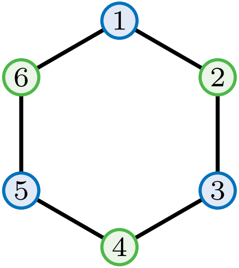

Motivation

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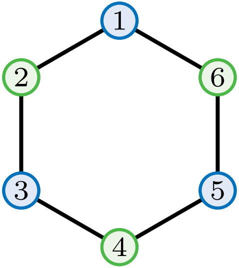

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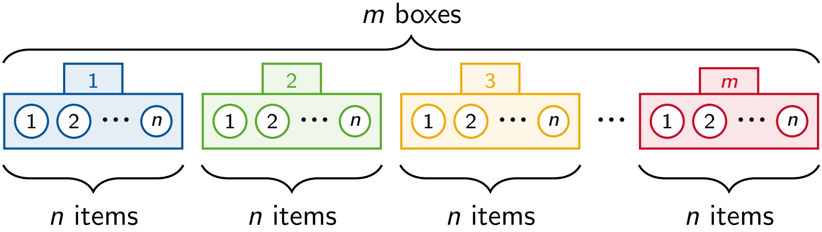

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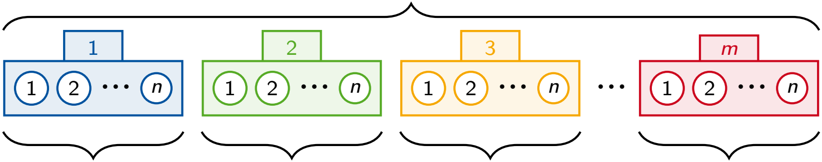

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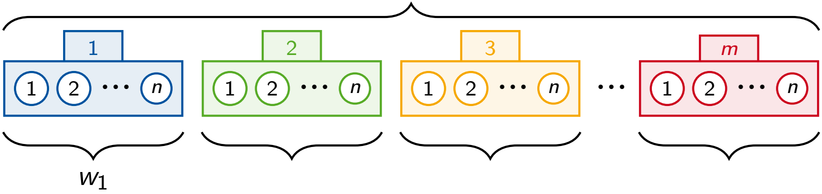

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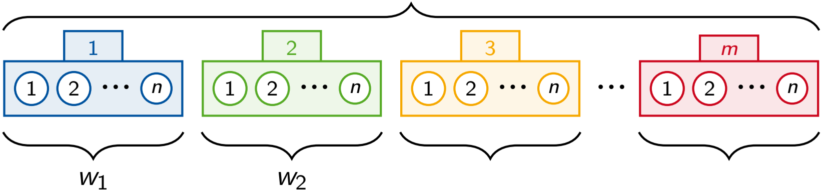

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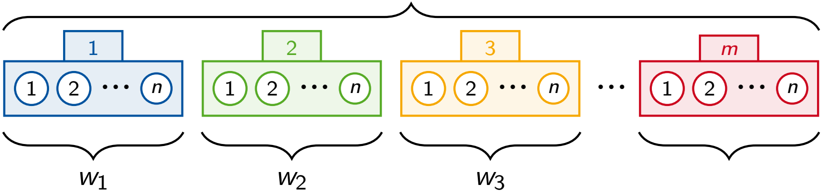

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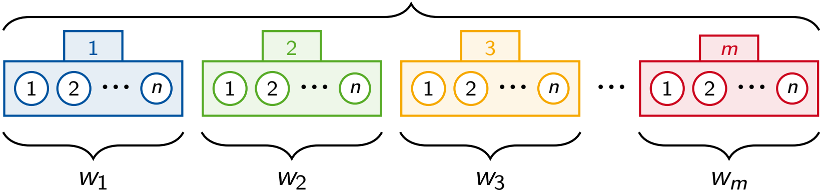

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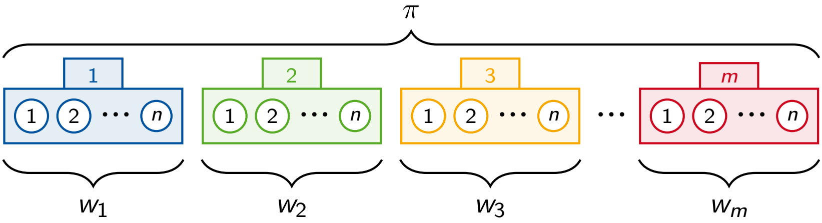

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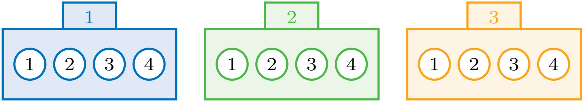
$$w := \Big(\, \overset{1}{(1,4)},\ \ \overset{2}{(1,2)(3,4)},\ \ \overset{3}{(1,2,3)}\,;\ \ \overset{\text{top}}{(2,3)} \,\Big) \in \mathrm{Sym}(4) \wr \mathrm{Sym}(3)$$



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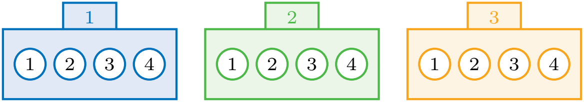

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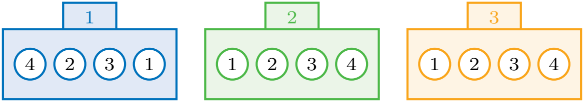

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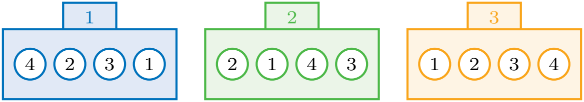

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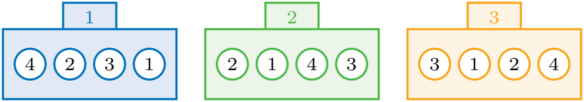

Wreath Cycle Decomposition

Definition: Territory

Let $W := K \wr \mathrm{Sym}(m)$ and $w := (w_1, \ldots, w_m; \pi) \in W$.
We define the **territory** of w as

$$\mathrm{terr}(w) := \mathrm{supp}(\pi) \cup \{1 \leq i \leq m : w_i \neq 1_K\}.$$

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Definition: Wreath Cycle

Let $W := K \wr \mathrm{Sym}(m)$ and $w := (w_1, \ldots, w_m; \pi) \in W$.
We call w a **wreath cycle** if either

- π is a non-trivial cycle and $\mathrm{terr}(w) = \mathrm{supp}(\pi)$; or
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$$u = \begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\ (& (), & (), & (), & (), & (), & (2,3), & (), & (); & (3,6) \;) \end{array}$$

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$$v = \begin{array}{c} \\ (\end{array} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\ (), & (), & (), & (1,3,4), & (), & (), & (), & (); & () \end{array} \begin{array}{c} \\ ) \end{array}$$

Definition: Wreath Cycle

Let $W := K \wr \mathrm{Sym}(m)$ and $w := (w_1, \ldots, w_m; \pi) \in W$.
We call w a **wreath cycle** if either

- π is a non-trivial cycle and $\mathrm{terr}(w) = \mathrm{supp}(\pi)$; or
- π is trivial and $|\mathrm{terr}(w)| = 1$.

$$u = \big(\overset{1}{()},\ \overset{2}{()},\ \overset{3}{()},\ \overset{4}{()},\ \overset{5}{()},\ \overset{6}{(2,3)},\ \overset{7}{()},\ \overset{8}{()};\ \overset{\text{top}}{(3,6)} \big)$$

$$v = \big(\overset{1}{()},\ \overset{2}{()},\ \overset{3}{()},\ \overset{4}{(1,3,4)},\ \overset{5}{()},\ \overset{6}{()},\ \overset{7}{()},\ \overset{8}{()};\ \overset{\text{top}}{()} \big)$$

Theorem

Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

$$w = \Big(\overset{1}{(1,2)(3,4)},\ \overset{2}{(3,4)},\ \overset{3}{()},\ \overset{4}{(1,3,4)},\ \overset{5}{()},\ \overset{6}{(2,3)},\ \overset{7}{(1,3)},\ \overset{8}{()};\ \overset{\text{top}}{(1,2)\ \ (3,6)} \Big)$$

Theorem

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Theorem

Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

$$w = \left(\begin{array}{ccccccccc} \boxed{1} & \boxed{2} & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\ (1,2)(3,4), & (3,4), & (), & (1,3,4), & (), & (2,3), & (1,3), & (); & \boxed{(1,2)}\ (3,6) \end{array} \right)$$

Theorem

Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

$$w = \big(\overset{1}{(1,2)(3,4)},\ \overset{2}{(3,4)},\ \overset{3}{()},\ \overset{4}{(1,3,4)},\ \overset{5}{()},\ \overset{6}{(2,3)},\ \overset{7}{(1,3)},\ \overset{8}{()};\ \overset{\text{top}}{(1,2)\ (3,6)} \big)$$

$$w_1 = \big(\overset{1}{(1,2)(3,4)},\ \overset{2}{(3,4)},\ \overset{3}{()},\ \overset{4}{()},\ \overset{5}{()},\ \overset{6}{()},\ \overset{7}{()},\ \overset{8}{()};\ \overset{\text{top}}{(1,2)} \big)$$

Theorem

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$$w_1 = \left(\begin{array}{cccccccccc} \overset{1}{(1,2)(3,4)}, & \overset{2}{(3,4)}, & \overset{3}{()}, & \overset{4}{()}, & \overset{5}{()}, & \overset{6}{()}, & \overset{7}{()}, & \overset{8}{()}; & \overset{\text{top}}{(1,2)} \end{array} \right)$$

Theorem

Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

$$
\begin{array}{rcccccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\
w & = (& (1,2)(3,4), & (3,4), & (), & (1,3,4), & (), & (2,3), & (1,3), & (); & (1,2)\ (3,6) \ ) \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\
w_1 & = (& (1,2)(3,4), & (3,4), & (), & (), & (), & (), & (), & (); & (1,2) \ ) \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\
w_2 & = (& (), & (), & (), & (), & (), & (2,3), & (), & (); & (3,6) \ )
\end{array}
$$

Theorem

Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

$$
\begin{array}{lcccccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\
w = (& & (1,2)(3,4), & (3,4), & (), & (1,3,4), & (), & (2,3), & (1,3), & (); & (1,2)\,(3,6) \;) \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\
w_1 = (& & (1,2)(3,4), & (3,4), & (), & (), & (), & (), & (), & (); & (1,2) \;) \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} \\
w_2 = (& & (), & (), & (), & (), & (), & (2,3), & (), & (); & (3,6) \;)
\end{array}
$$

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Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

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$$w_3 = \big(\overset{1}{()},\ \overset{2}{()},\ \overset{3}{()},\ \overset{4}{(1,3,4)},\ \overset{5}{()},\ \overset{6}{()},\ \overset{7}{()},\ \overset{8}{()};\ \overset{\text{top}}{()} \big)$$

Wreath Cycle Decomposition

Theorem

Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

$$\begin{array}{rcccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} & \\
w = (& (1,2)(3,4), & (3,4), & (), & (1,3,4), & (), & (2,3), & (1,3), & (); & (1,2)\;(3,6) &) \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} & \\
w_1 = (& (1,2)(3,4), & (3,4), & (), & (), & (), & (), & (), & (); & (1,2) &) \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} & \\
w_2 = (& (), & (), & (), & (), & (), & (2,3), & (), & (); & (3,6) &) \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{top} & \\
w_3 = (& (), & (), & (), & (1,3,4), & (), & (), & (), & (); & () &)
\end{array}$$

Theorem

Let $W := K \wr \mathrm{Sym}(m)$ and $w \in W$. Then w can be written as a product of wreath cycles with pairwise disjoint territory.

$$w = \big(\overset{1}{(1,2)(3,4)},\ \overset{2}{(3,4)},\ \overset{3}{()},\ \overset{4}{(1,3,4)},\ \overset{5}{()},\ \overset{6}{(2,3)},\ \overset{7}{(1,3)},\ \overset{8}{()};\ \overset{\text{top}}{(1,2)\,(3,6)}\big)$$

$$w_1 = \big(\overset{1}{(1,2)(3,4)},\ \overset{2}{(3,4)},\ \overset{3}{()},\ \overset{4}{()},\ \overset{5}{()},\ \overset{6}{()},\ \overset{7}{()},\ \overset{8}{()};\ \overset{\text{top}}{(1,2)}\big)$$

$$w_2 = \big(\overset{1}{()},\ \overset{2}{()},\ \overset{3}{()},\ \overset{4}{()},\ \overset{5}{()},\ \overset{6}{(2,3)},\ \overset{7}{()},\ \overset{8}{()};\ \overset{\text{top}}{(3,6)}\big)$$

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$$w_4 = \big(\overset{1}{()},\ \overset{2}{()},\ \overset{3}{()},\ \overset{4}{()},\ \overset{5}{()},\ \overset{6}{()},\ \overset{7}{(1,3)},\ \overset{8}{()};\ \overset{\text{top}}{()}\big)$$

Results from [Bernhardt, Niemeyer, R., Wollenhaupt, '22]

Let $W := K \wr H$ and $H \leq \mathrm{Sym}(m)$. We describe algorithms ...

- to solve the **Conjugacy Problem** for two elements in W;
- to compute the **Centraliser** of an element in W; and
- to compute all **Conjugacy Classes** of elements in W.

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Main Idea

Break down problems ...

- from wreath product elements onto wreath cycles; and
- from W onto K and H.

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Main Idea

Break down problems ...

- from wreath product elements onto wreath cycles; and
- from W onto K and H.

GAP **Session**

Display:

```
gap> LoadPackage("WPE");;
gap> M11 := MathieuGroup(11);;
gap> S5 := SymmetricGroup(5);;
gap> G := WreathProduct(M11, S5);;
gap> g := PseudoRandom(G);
(1,53,10,45,5,49,11,52,7,46,4,55,6,47,9,51,3,50,8,54,2,48)(12,
19,13)(14,18,22)(16,20,21)(23,41,30,34)(24,36,29,39,28,37,32,
38,26,44,25,43)(27,40,31,42,33,35)
gap> iso := IsomorphismWreathProduct(G);;
gap> opts := rec(horizontal := false, labelColor := "blue");;
gap> Display(g ^ iso, opts);
  1: (1,9,7,2,4,11,8,10)(3,6)
  2: (1,8,2)(3,7,11)(5,9,10)
  3: (1,8)(2,3,10,5,7,6,4,11)
  4: (2,5,4,10)(3,7,9,11)
  5: (1,5,11,6,8,7,3,9,10,2,4)
top: (1,5)(3,4)
```

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top: (1,5)(3,4)
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  5: (1,5,11,6,8,7,3,9,10,2,4)
top: (1,5)(3,4)
```

Conjugacy Problem:

```
gap> LoadPackage("WPE");;
gap> M11 := MathieuGroup(11);;
gap> S100 := SymmetricGroup(100);;
gap> G := WreathProduct(M11, S100);;
gap> Length(String(Size(G)));
548
gap> NrMovedPoints(G);
1100
gap> g := PseudoRandom(G);;
gap> h := g ^ PseudoRandom(G);;
gap> c := RepresentativeAction(G, g, h);; # 10 ms
gap> g ^ c = h;
true
```

Conjugacy Problem:

```
gap> LoadPackage("WPE");;
gap> M11 := MathieuGroup(11);;
gap> S100 := SymmetricGroup(100);;
gap> G := WreathProduct(M11, S100);;
gap> Length(String(Size(G)));
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gap> NrMovedPoints(G);
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gap> g ^ c = h;
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Conjugacy Problem:

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true
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gap> c := RepresentativeAction(G, g, h);; # 10 ms
gap> g ^ c = h;
true
```

Conjugacy Classes:

```
gap> LoadPackage("WPE");;
gap> M11 := MathieuGroup(11);;
gap> S10 := SymmetricGroup(10);;
gap> G := WreathProduct(M11, S10);;
gap> C := ConjugacyClasses(G);; # 20 s
gap> Size(C);
1605340
```

Conjugacy Classes:

```
gap> LoadPackage("WPE");;
gap> M11 := MathieuGroup(11);;
gap> S10 := SymmetricGroup(10);;
gap> G := WreathProduct(M11, S10);;
gap> C := ConjugacyClasses(G);; # 20 s
gap> Size(C);
1605340
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1605340
```

Group	GAP4	Magma	WPE	#Conjugacy classes
$S_4 \wr S_8$	60 s	4 s	$<$ 1 s	6 765
$\mathrm{SU}(3,2) \wr A_7$	36 m	5 m	22 s	398 592
$M_{24} \wr S_7$	$>$ 24 h	error	160 s	9 293 050
$S_7 \wr \mathrm{PSL}(2,7)$	$>$ 24 h	30 m	5 m	15 342 750

Thank you for your attention!