SOTGrps—Extending the Small Groups database

GAP Days Summer 2022, RWTH Aachen

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An MPhil project supervised by Heiko Dietrich

Some known results

Let p, q, r be distinct primes.

- **1893**: Hölder determined groups of order p^3, p^4, p^2q, pqr.
- **1893**: Cole & Glover determined groups whose orders factorise into three primes.
- **1895**: Hölder classified groups of squarefree order.
- **1898**: Bagnera determined the groups of order p^5.
- **1902**: La Vavasseur determined groups of order p^2q^2.
- **1903**: La Vavasseur determined groups of order $16p$ for odd prime p.
- **1906**: Glenn studied groups of order p^2qr.
- **1909**: Tripp determined groups of order p^3q^2.
- **1919**: Nyhlén determined groups of order $16p^2$ and $8p^3$ for odd prime p.
- **1934**: Lunn & Senior determined groups of $16p$ and $32p$.
- **1977**: Western determined groups of order p^3q.
- **1980**: James determined the groups of order p^6 for odd p.
- **1982**: Laue enumerated groups of odd order p^aq^b where $a+b \leq 6$ and $a, b < 5$.
- **1990**: Newman & O'Brien derived the p-group generation algorithm.
- **2005**: O'Brien & Vaughan-Lee enumerated groups of order p^7 for odd p.
- **2005**: Dietrich & Eick developed a construction algorithm for cubefree groups.
- **2007**: Slattery developed an algorithm for squarefree groups.
- **2017**: Eick enumerated groups whose orders factorise into at most four primes.
- **2018**: Eick & Moede enumerated groups of order p^nq for $n \leq 5$.

In the digital era

The Small Groups library contains the following groups:

- those of order at most 2000 except 1024;
- those of cubefree order at most 50 000;
- those of order p^7 for the primes p = 3,5,7,11;
- those of order p^n for $n \leq 6$ and all primes p;
- those of order $q^n p$ for q^n dividing 2^8, 3^6, 5^5 or 7^4 and all primes p with $p \neq q$;
- those of squarefree order;
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- those of squarefree order;
- those whose order factorise into at most 3 primes.

For example,

```
gap> SmallGroupsAvailable(997^6);
true
gap> NumberSmallGroups(997^6);
3021372
gap> SmallGroup(997^6,1000);
<pc group of size 982134461213542729 with 6 generators>
```

Groups of small order type—an MPhil project

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Main results

For the groups whose orders factorise into at most 4 primes (or of order p^4q), we give

- new determination of groups (explicit group presentations);
- new counting formulas for enumeration;
- new algorithms and GAP implementation for ID-functionality.

Example: constructing groups of order p^2q

Let Δ_y^x be the Kronecker divisibility delta: $\Delta_y^x = 1$ if $x \mid y$ and $\Delta_y^x = 0$ otherwise.

Groups of order p^2q.

PC-relators	Parameters	Number of groups
Cluster 1: nilpotent		
a^{p^2q}		1
a^{pq}, b^p		1
Cluster 2: non-nilpotent, normal C_p^2		
$a^q, b^p, c^p, b^a/b^{\rho(p,q)}$		Δ_{p-1}^q
$a^q, b^p, c^p, (b^a, c^a)/(b,c)^{M_2(p,q,\sigma_q^k)}$	$0 \leq k \leq \lfloor \frac{1}{2}(q-1) \rfloor$	$\frac{1}{2}(q+1-\Delta_q^2)\Delta_{p-1}^q$
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For each cluster, construct a “canonically” ordered list of isomorphism types with polycyclic presentations.

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For each cluster, construct a "canonically" ordered list of isomorphism types with polycyclic presentations.

Example: identifying groups of order p^2q

Let H be the permutation group generated by

$$\{(2,29,23,15,9,5,3)(4,27,24,16,10,8,7)(6,20,12,25,17,18,11)(13,28,22,26,19,14,21)$$

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1. **Determine the order and order type of H**: $n = |H| = 29^2 \cdot 7$.
2. **Compute a Sylow subgroup for each prime divisor of n**: let $P \in \mathrm{Syl}_{29}(H)$ and $Q \in \mathrm{Syl}_7(H)$.
3. **Whether H contains a normal Sylow subgroup**: $P \trianglelefteq H$, so H is a split extension of Q by N.
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6. **Determine the position of H in the corresponding Cluster** with respect to the canonical ordering: We find that u acts on v, w via $M = \left(\begin{smallmatrix} 7 & 0 \\ 0 & 16 \end{smallmatrix}\right) \in \mathrm{GL}_2(29)$. This matrix has eigenvalues $\{7, 16\}$, and $\mathrm{diag}(7,16) = \mathrm{diag}(a^3, a)$ with $a = \rho(29,7) = 16$. Since $\langle M \rangle$ is conjugate to $\langle \mathrm{diag}(a, a^3) \rangle$ and $\sigma_7 = 3$, this determines the parameter $k = 1$ in the canonical form.
7. **Determine the group ID**: H has ID $(29^2 \cdot 7, 5)$.

A joint work with Dietrich & Eick (2021)

Our enumeration results coincide with those in Eick's 2017 arXiv paper.

Theorem 2.1. *Let p, q, and r be distinct primes.*

a) **Order p^2q:**

- $\mathcal{N}(p^2q) = 5$ *for* $q = 2$.
- $\mathcal{N}(p^2q) = 2 + \frac{q+5}{2}\Delta^q_{p-1} + \Delta^q_{p+1} + 2\Delta^p_{q-1} + \Delta^{p^2}_{q-1}$ *for* $q > 2$.

b) **Order p^3q:**

- *There are two special cases* $\mathcal{N}(2^3.3) = 15$ *and* $\mathcal{N}(2^3.7) = 13$.
- $\mathcal{N}(p^3q) = 15$ *if* $q = 2$.
- $\mathcal{N}(p^3q) = 12 + 2\Delta^4_{q-1} + \Delta^8_{q-1}$ *if* $p = 2$ *and* $q \notin \{3, 7\}$.
- *If p and q are both odd, then*

$$\begin{aligned}\mathcal{N}(p^3q) \quad &= \quad 5 + \tfrac{q^2+13q+36}{6}\Delta^q_{p-1} + (p+5)\Delta^p_{q-1} + \tfrac{2}{3}\Delta^3_{q-1}\Delta^q_{p-1} \\ &\quad + \Delta^q_{(p+1)(p^2+p+1)}(1 - \Delta^q_{p-1}) + \Delta^q_{p+1} + 2\Delta^{p^2}_{q-1} + \Delta^{p^3}_{q-1}.\end{aligned}$$

c) **Order p^2q^2 with $p > q$:**

- *There is one special case* $\mathcal{N}(2^2.3^2) = 14$.
- $\mathcal{N}(p^2q^2) = 12 + 4\Delta^4_{p-1}$ *if* $q = 2$ *and* $p \neq 3$.
- $\mathcal{N}(p^2q^2) = 4 + \frac{1}{2}(q^2 + q + 4)\Delta^{q^2}_{p-1} + (q+6)\Delta^q_{p-1} + 2\Delta^q_{p+1} + \Delta^{q^2}_{p+1}$ *if* $q > 2$.

d) **Order p^2qr with $q < r$:**

- *There is one special case* $\mathcal{N}(2^2.3.5) = 13$.
- $\mathcal{N}(p^2qr) = 10 + (2r+7)\Delta^r_{p-1} + 3\Delta^r_{p+1} + 6\Delta^p_{r-1} + 2\Delta^{p^2}_{r-1}$ *if* $q = 2$.
- *If* $q > 2$ *and* $(p, q, r) \neq (2, 3, 5)$, *then*

$$\begin{aligned}\mathcal{N}(p^2qr) \quad &= \quad 2 \;+\; (p^2-p)\Delta^{p^2}_{q-1}\Delta^{p^2}_{r-1} \;+\; \Delta^r_{p+1} \;+\; \Delta^q_{p+1} \;+\; \Delta^{p^2}_{r-1} \;+\; \Delta^{p^2}_{q-1} \\ &\quad + (p-1)\big(\Delta^{p^2}_{q-1}\Delta^p_{r-1} \;+\; \Delta^{p^2}_{r-1}\Delta^p_{q-1} \;+\; 2\Delta^p_{r-1}\Delta^p_{q-1}\big) \\ &\quad + \Delta^p_{r-1}\Delta^p_{q-1} \;+\; \Delta^p_{r-1}\Delta^q_{p-1} \;+\; \tfrac{1}{2}(q-1)(q+4)\Delta^q_{p-1}\Delta^q_{r-1} \\ &\quad + \tfrac{1}{2}(q-1)\big(\Delta^q_{p+1}\Delta^q_{r-1} \;+\; \Delta^q_{p-1} \;+\; \Delta^{qr}_{p-1} \;+\; 2\Delta^{pq}_{r-1}\Delta^q_{p-1}\big) \\ &\quad + \tfrac{1}{2}(qr+1)\Delta^{qr}_{p-1} \;+\; \tfrac{1}{2}(r+5)\Delta^r_{p-1}(1+\Delta^q_{p-1}) \;+\; 2\Delta^q_{r-1} \\ &\quad + \Delta^{qr}_{p^2-1} \;+\; 2\Delta^{pq}_{r-1} \;+\; \Delta^{p^2q}_{r-1} \;+\; 2\Delta^p_{q-1} \;+\; 3\Delta^q_{p-1} \;+\; 2\Delta^p_{r-1}.\end{aligned}$$

SOTGrps in action

For a group G of order that is `SOTGroupIsAvailable`, `SOTGrps` provides the following functions:

- `NumberOfSOTGroups`(n).
- `AllSOTGroups`(n).
- `SOTGroup`(n, k).
- `IdSOTGroup`(G).

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For example, groups of order 2662 are covered in the dynamic database of `SOTGrps`.

```
gap> NumberOfSOTGroups(2662);
15
gap> AllSOTGroups(2662);
[ <pc group of size 2662 with 4 generators>, <pc group of size 2662 with 4 generators>, <pc group of size 2662 with 4 generators>,
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  <pc group of size 2662 with 4 generators>, <pc group of size 2662 with 4 generators>, <pc group of size 2662 with 4 generators> ]
```

It constructs groups of order $n \in \mathcal{O}$ on demand.

```
gap> NumberOfSOTGroups(59719399^2*859);
434
gap> AllSOTGroups(59719399^2*859);;time;
411
gap> NumberOfSOTGroups(607*3643^3);
62735
gap> SOTGroup(607*3643^3,50000);time;
<pc group of size 29347168445149 with 4 generators>
344
```

Practical performances and bottlenecks

74844 groups of order p^2q, p^3q, p^2q^2, p^2qr at most 50000

- **Construction.** SmallGroups[1] 27359 secs vs SOTGrps 47 secs
  GrpConst[2] 150 secs vs SOTGrps 0.2 secs
- **Identification.** SmallGroups 43356 secs vs SOTGrps 259 secs

Groups of order $11^4 \cdot 5 = 73205$

- **Construction.** GrpConst 326 secs vs SOTGrps 0.07 secs
- **Identification.** GrpConst unavailable vs SOTGrps 0.9 secs

[1] Only for those 74562 groups available in SmallGroups.
[2] Only for the remaining 282 groups unavailable in SmallGroups.

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In general, we face increasing difficulties when the prime factors become large.

For example,

- arithmetic in polycyclic groups (collection) seems to be slow for large primes;
- we encountered a bug in the `LogFFE` function along the way.

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[2] Only for the remaining 282 groups unavailable in SmallGroups.

Thank you!

Background theory and main approach

Well-known results:

- Classification of group extensions; relations between equivalence classes and 2-cohomology groups.
- Classification of strong isomorphism classes of group extensions and compatible pairs.
- Isomorphism classes of semidirect products (Taunt 1955).
- Finite groups with cyclic Sylow subgroups are solvable.
- For distinct primes p, q, Burnside's theorem asserts that the finite groups of order $p^a q^b$ are solvable (1904).
- The odd-order theorem (proved by Feit and Thompson in 1962) states that all groups of odd order are solvable.

Constructing solvable groups by iterating group extensions

- We construct p-groups of order dividing p^4 as extensions with abelian kernels.
- For groups of order $p^a q^b$ we make a case distinction on the existence of normal Sylow subgroups and further divide the cases by the isomorphism type of Sylow subgroups.
- For other groups whose orders contains more than two factors, we make a case distinction on the Fitting subgroups.