GAP Package - LINS

Low Index Normal Subgroups

Friedrich Rober

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Overview

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(**L**ow **I**ndex **N**ormal **S**ubgroups)
provides an algorithm for computing the normal subgroups of a finitely presented group up to some given index bound.

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  (based on work of Derek Holt and David Firth)
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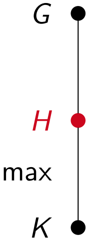
Used by some researchers in the area of
error correcting codes for quantum computers.

General Approach

Quotients

Idea 1: Iteration over computed normal subgroups

Let G be a group, and $H \trianglelefteq G$.
Search for all maximal G-normal subgroups under H.



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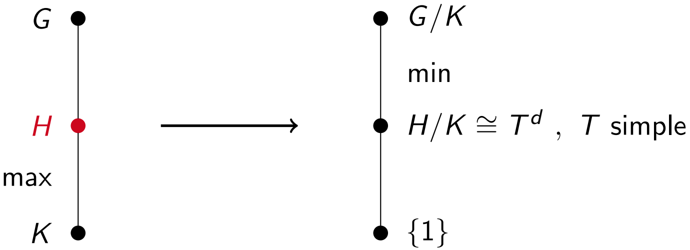

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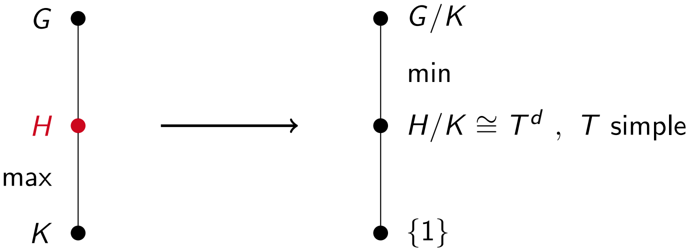

- T abelian $\Longrightarrow$ P-Quotient, $T \cong C_p$
- T non-abelian $\Longrightarrow$ T-Quotient, CFSG

Idea 2: Reduce the explicit search for quotients

Let G be a group, and $K, H \trianglelefteq G$. Then $U := K \cap H \trianglelefteq G$.
Do not search for U as a quotient of some supergroup,
but rather compute it as an intersection.

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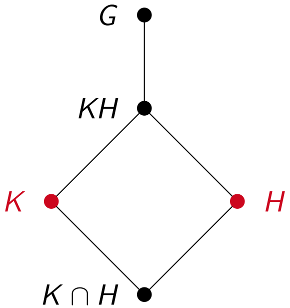

$$[G : K \cap H] = [G : KH] \cdot [KH : K] \cdot [KH : H]$$

Idea:

- Only a few T-Quotients cannot be computed as intersections.
- We can find these subgroups as quotients under G.
- Use pre-computed list of these quotients and GQuotients.

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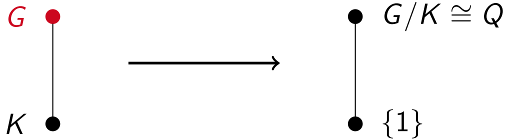

- T non-abelian
- $T^d \cong \mathrm{Inn}(T^d) \leq Q \leq \mathrm{Aut}(T^d) \cong \mathrm{Aut}(T) \wr \mathrm{Sym}(d)$
- Q acts transitively on socle factors $\{T_1, \ldots, T_d\}$

P-Quotients

Idea:

- P-Quotients are e.a. p-quotients for some prime p.
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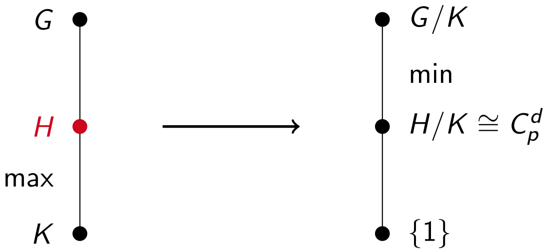

Approach

- Technical but cheap checks on p to decide, if p-quotients cannot be computed as intersections (pre-computed Schur-Multipliers of T^d, T non-abelian simple)
- Compute class-1 p-quotient under H, say M ($\mathbb{F}_p$-submodule $\iff$ e.a. p-quotient under H)
- Construct G-action on M via conjugation ($\mathbb{F}_pG$-submodule $\iff$ above & G-normal subgroup)
- Use MeatAxe to compute maximal $\mathbb{F}_pG$-submodules of M.

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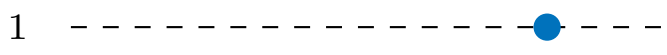

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$$G = \mathrm{Alt}(5) \times C_2 \times C_3$$

up to index $n = 200\,.$

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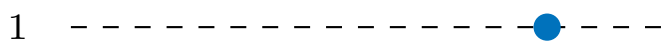

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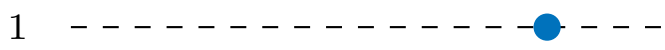

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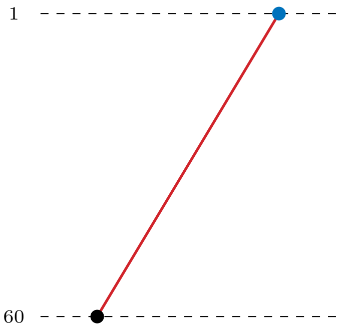

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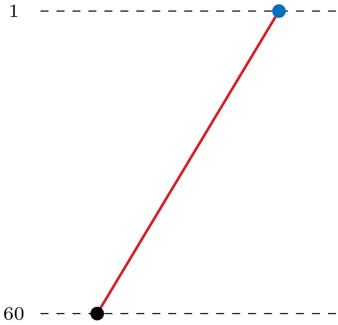

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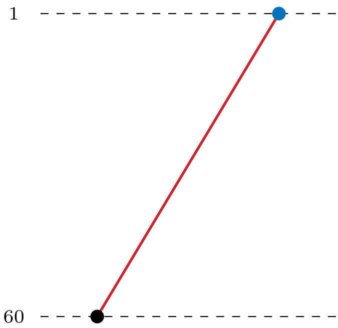

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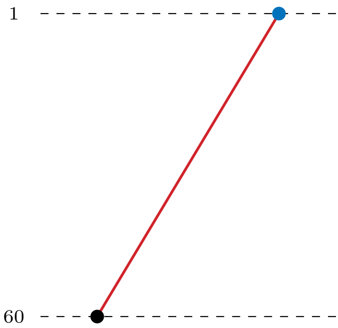

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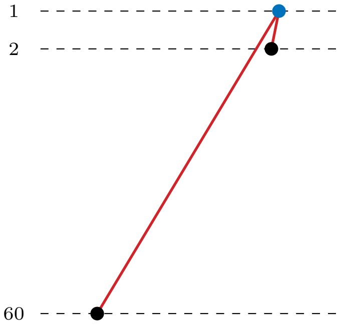

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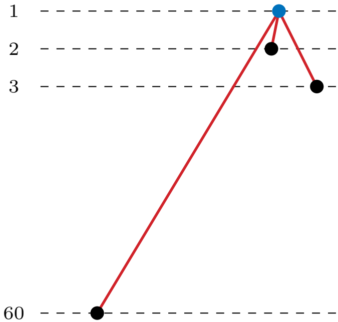

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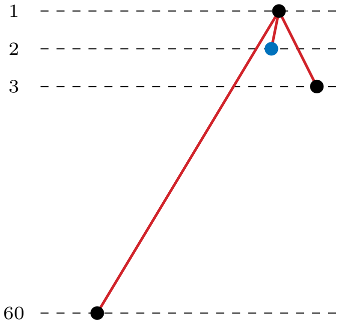

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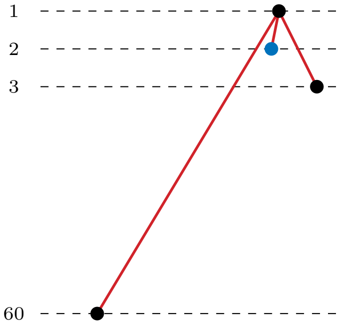

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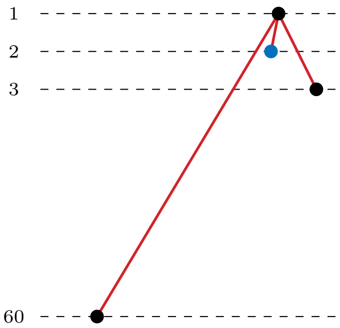

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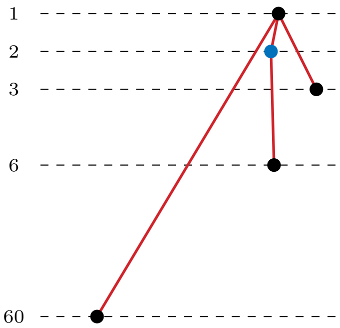

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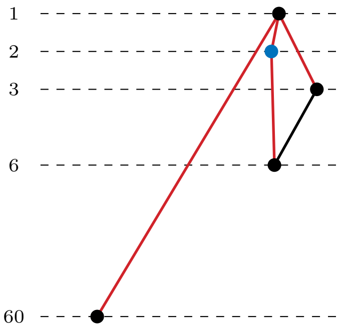

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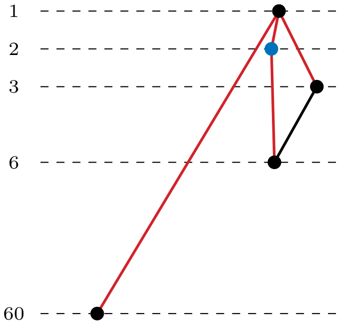

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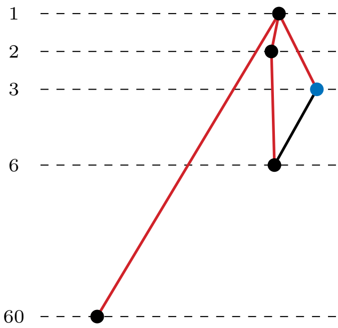

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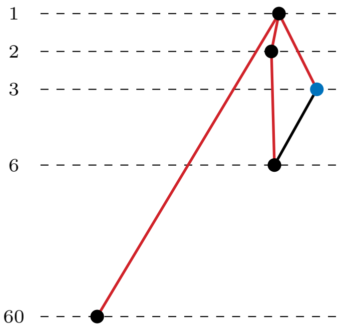

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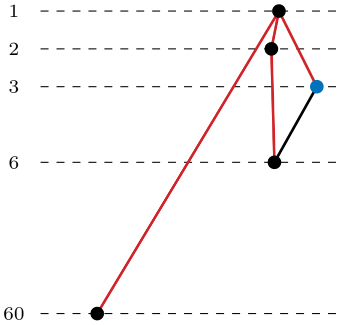

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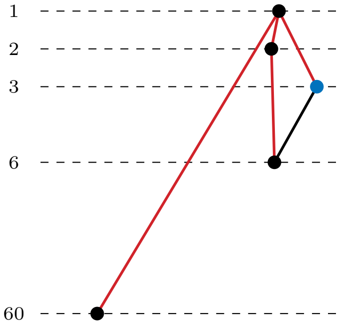

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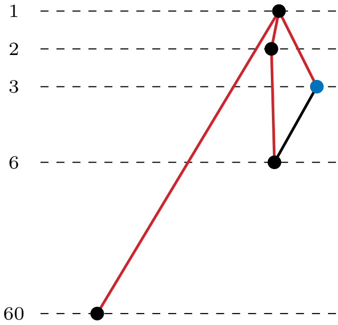

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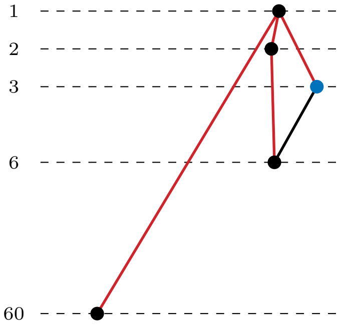

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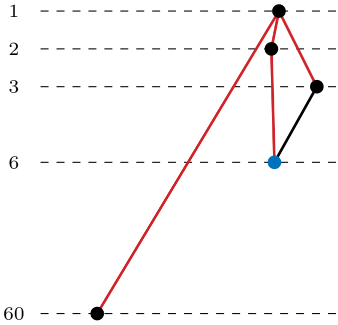

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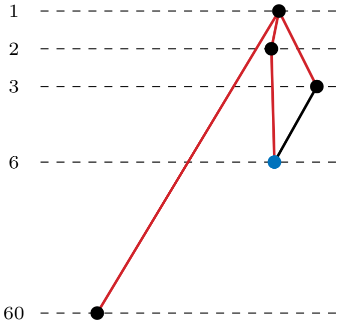

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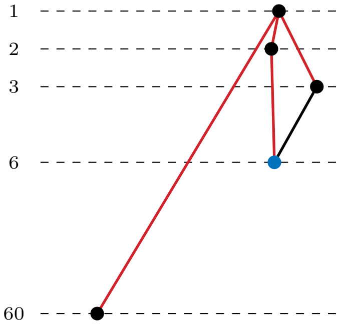

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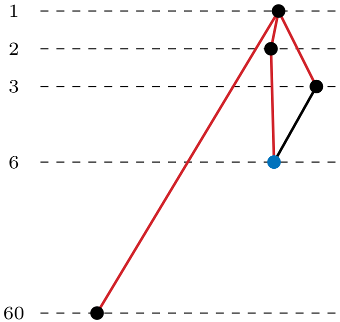

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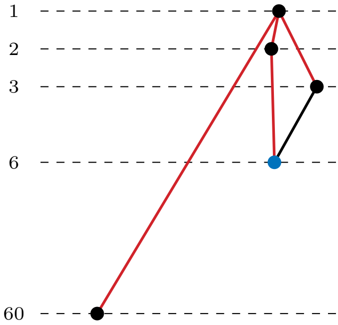

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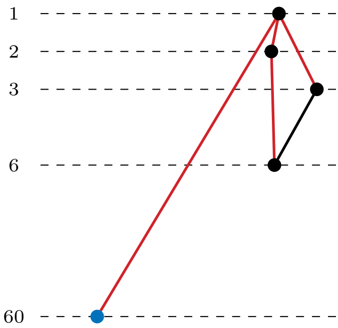

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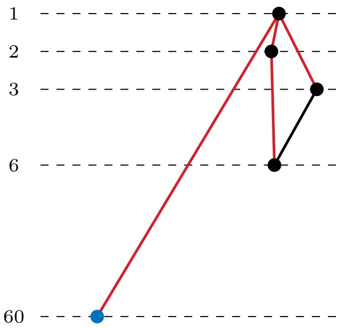

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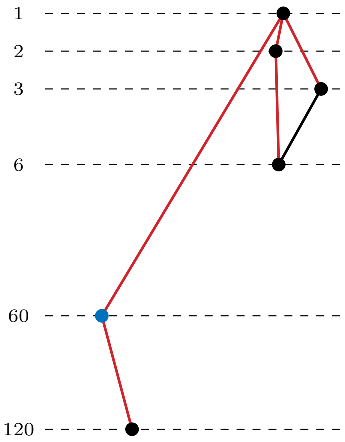

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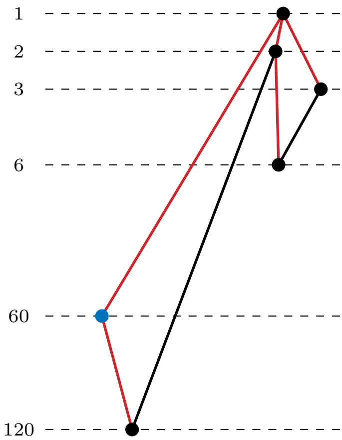

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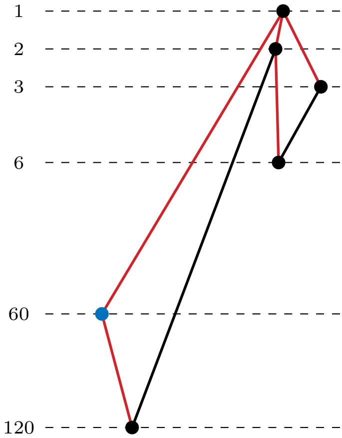

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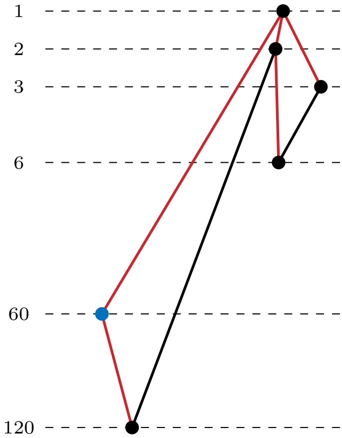
⋮	⋮



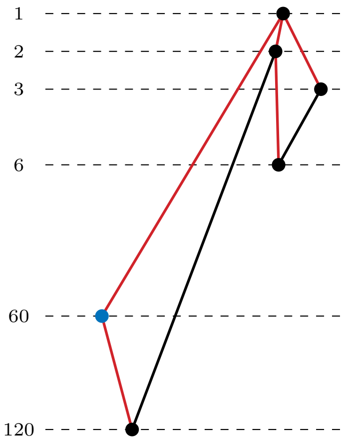
Compute all normal subgroups of

$$G = \mathrm{Alt}(5) \times C_2 \times C_3$$

up to index $n = 200$.

Intersections under H:

$[G : K]$	$[G : K \cap H]$	Compute?
2	120	✗
3	180	✓
6	360	✗



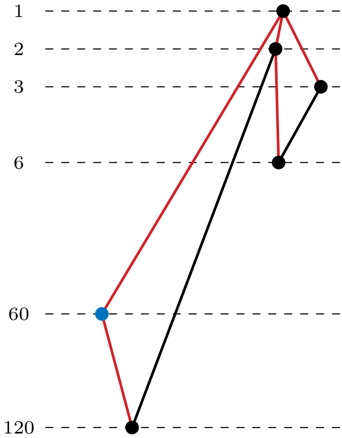
Compute all normal subgroups of

$$G = \mathrm{Alt}(5) \times C_2 \times C_3$$

up to index $n = 200$.

Intersections under H:

$[G : K]$	$[G : K \cap H]$	Compute?
2	120	✗
3	180	✓
6	360	✗



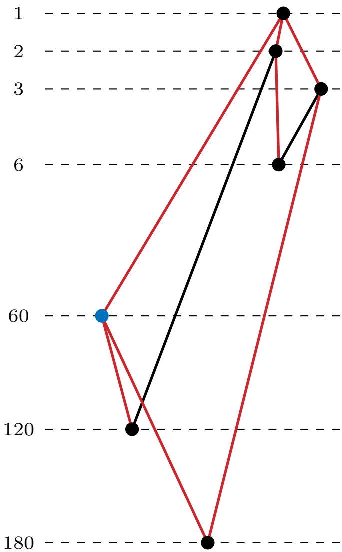
Compute all normal subgroups of

$$G = \mathrm{Alt}(5) \times C_2 \times C_3$$

up to index $n = 200$.

Intersections under H:

$[G : K]$	$[G : K \cap H]$	Compute?
2	120	✗
3	180	✓
6	360	✗



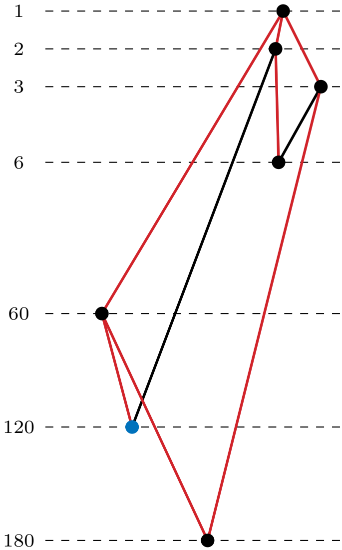
Compute all normal subgroups of

$$G = \mathrm{Alt}(5) \times C_2 \times C_3$$

up to index $n = 200$.

Intersections under H:

$[G : K]$	$[G : K \cap H]$	Compute?
2	120	✗
3	180	✓
6	360	✗



Compute all normal subgroups of

$$G = \mathrm{Alt}(5) \times C_2 \times C_3$$

up to index $n = 200$.

Terminate Search!

GAP **Session**

Normal Subgroups of Dih(50):

```
gap> LoadPackage("LINS");;
gap> n := 50;;
gap> G := DihedralGroup(n);;
gap> gr := LowIndexNormalSubgroupsSearchForAll(G, n);
<lins graph found 4 normal subgroups up to index 50>
gap> L := List(gr);
[ <lins node of index 1>, <lins node of index 2>,
  <lins node of index 10>, <lins node of index 50> ]
gap> IsoTypes := List(L, node -> StructureDescription(Grp(node)));
[ "D50", "C25", "C5", "1" ]
```

Normal Subgroups of Dih(50):

```
gap> LoadPackage("LINS");;
gap> n := 50;;
gap> G := DihedralGroup(n);;
gap> gr := LowIndexNormalSubgroupsSearchForAll(G, n);
<lins graph found 4 normal subgroups up to index 50>
gap> L := List(gr);
[ <lins node of index 1>, <lins node of index 2>,
  <lins node of index 10>, <lins node of index 50> ]
gap> IsoTypes := List(L, node -> StructureDescription(Grp(node)));
[ "D50", "C25", "C5", "1" ]
```

Normal Subgroups of Dih(50):

```
gap> LoadPackage("LINS");;
gap> n := 50;;
gap> G := DihedralGroup(n);;
gap> gr := LowIndexNormalSubgroupsSearchForAll(G, n);
<lins graph found 4 normal subgroups up to index 50>
gap> L := List(gr);
[ <lins node of index 1>, <lins node of index 2>,
  <lins node of index 10>, <lins node of index 50> ]
gap> IsoTypes := List(L, node -> StructureDescription(Grp(node)));
[ "D50", "C25", "C5", "1" ]
```

WIP: Normal Subgroup of $C_3^2 \times C_4 \times C_5$ of Index 15:

```
gap> LoadPackage("LINS");;
gap> pList := [3, 3, 4, 5];;
gap> G := DirectProduct(List(pList, p -> CyclicGroup(p)));;
gap> gr := LowIndexNormalSubgroupsSearchForIndex(G, 15, 1);
<lins graph found 7 normal subgroups up to index 15>
gap> L := ComputedNormalSubgroups(gr);
[ <lins node of index 15> ]
gap> IsoTypes := List(L, node -> StructureDescription(Grp(node)));
[ "C12" ]
gap> gr2 := LowIndexNormalSubgroupsSearchForAll(G, 15);
<lins graph found 22 normal subgroups up to index 15>
gap> Number(List(gr2), rH -> Index(rH) = 15);
4
```

WIP: Normal Subgroup of $C_3^2 \times C_4 \times C_5$ of Index 15:

```
gap> LoadPackage("LINS");;
gap> pList := [3, 3, 4, 5];;
gap> G := DirectProduct(List(pList, p -> CyclicGroup(p)));;
gap> gr := LowIndexNormalSubgroupsSearchForIndex(G, 15, 1);
<lins graph found 7 normal subgroups up to index 15>
gap> L := ComputedNormalSubgroups(gr);
[ <lins node of index 15> ]
gap> IsoTypes := List(L, node -> StructureDescription(Grp(node)));
[ "C12" ]
gap> gr2 := LowIndexNormalSubgroupsSearchForAll(G, 15);
<lins graph found 22 normal subgroups up to index 15>
gap> Number(List(gr2), rH -> Index(rH) = 15);
4
```

WIP: Normal Subgroup of $C_3^2 \times C_4 \times C_5$ of Index 15:

```
gap> LoadPackage("LINS");;
gap> pList := [3, 3, 4, 5];;
gap> G := DirectProduct(List(pList, p -> CyclicGroup(p)));;
gap> gr := LowIndexNormalSubgroupsSearchForIndex(G, 15, 1);
<lins graph found 7 normal subgroups up to index 15>
gap> L := ComputedNormalSubgroups(gr);
[ <lins node of index 15> ]
gap> IsoTypes := List(L, node -> StructureDescription(Grp(node)));
[ "C12" ]
gap> gr2 := LowIndexNormalSubgroupsSearchForAll(G, 15);
<lins graph found 22 normal subgroups up to index 15>
gap> Number(List(gr2), rH -> Index(rH) = 15);
4
```

WIP: Normal Subgroup of $C_3^2 \times C_4 \times C_5$ of Index 15:

```
gap> LoadPackage("LINS");;
gap> pList := [3, 3, 4, 5];;
gap> G := DirectProduct(List(pList, p -> CyclicGroup(p)));;
gap> gr := LowIndexNormalSubgroupsSearchForIndex(G, 15, 1);
<lins graph found 7 normal subgroups up to index 15>
gap> L := ComputedNormalSubgroups(gr);
[ <lins node of index 15> ]
gap> IsoTypes := List(L, node -> StructureDescription(Grp(node)));
[ "C12" ]
gap> gr2 := LowIndexNormalSubgroupsSearchForAll(G, 15);
<lins graph found 22 normal subgroups up to index 15>
gap> Number(List(gr2), rH -> Index(rH) = 15);
4
```

Runtime Comparison

Group	Index	Magma	LINS	#Normal Subgroups
$C_2 * C_5$	1 000	5 s	8 s	88
$C_2 * C_5$	3 000	88 s	200 s	295
$C_2 * C_8$	2 000	12 h	7 h	8 092
$\mathbb{Z}^2 \rtimes C_3$	1 000	104 s	7 s	206
$\Delta(3,3,3)$	10 000	5 m	22 s	68
Dih(1 000)	1 000	1 s	4 s	15
Dih(10 000)	10 000	26 s	14 m	23
$\mathrm{Alt}(5)^3$	216 000	39 m	2 m	8

Runtime Comparison of Sub-Algorithms

Disclaimer:

These observations should not be taken too seriously!

Sub-Algorithm	Magma	GAP4
IsomorphismFpGroup	👎	👍
GQuotients	👍	👎
MeatAxe	👍	👎
Intersection	👎	👍

Thank you for your attention!