An algorithm for constructing all supercharacter theories of a finite group

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Motivation

Hendrickson make explicit the connection between supercharacter theories and **Schur rings**, and they provide supercharacter theory constructions which correspond to Schur ring products.

A. O. F. Hendrickson, Supercharacter theory constructions corresponding to Schur ring products, *Comm. Algebra* **40**(12) (2012) 4420–4438.

Proposition 2.4. *Let G be a finite group. Then there is a bijection*

$$\left\{\begin{array}{c}\textit{supercharacter theories}\\ (\mathscr{X},\mathscr{K})\ \textit{of}\ G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}S\textit{-rings over}\ G\\ \textit{contained in}\ \mathbf{Z}(\mathbb{C}[G])\end{array}\right\}$$

$$(\mathscr{X},\mathscr{K}) \longmapsto \operatorname{span}_{\mathbb{C}}\{\widehat{K} : K \in \mathscr{K}\}.$$

In 2008, P. Diaconis and I. M. Isaacs introduced the theory of supercharacters axiomatically, building upon seminal work of C. Andre on the representation theory of unipotent matrix groups [1], [2]. Supercharacter techniques have been used to study the **Hopf algebra of symmetric functions of noncommuting variables** [3], **random walks on upper triangular matrices** [4], In 2008, P. Diaconis and I. M. Isaacs introduced the theory of supercharacters axiomatically, building upon seminal work of C. Andre on the representation theory of unipotent matrix groups [1], [2]. Supercharacter techniques have been used to study the **Hopf algebra of symmetric functions of noncommuting variables** [3],**random walks on upper triangular matrices** [4],**combinatorial properties of Schur rings** [5], [6], [7], In 2008, P. Diaconis and I. M. Isaacs introduced the theory of supercharacters axiomatically, building upon seminal work of C. Andre on the representation theory of unipotent matrix groups [1], [2]. Supercharacter techniques have been used to study the **Hopf algebra of symmetric functions of noncommuting variables** [3],**random walks on upper triangular matrices** [4],**combinatorial properties of Schur rings** [5], [6], [7],and **Ramanujan sums** [8].

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5 Persi Diaconis and Nathaniel Thiem, Supercharacter formulas for pattern groups, Trans. Amer. Math. Soc., 361(7):3501-3533, 2009.

6 Nathaniel Thiem, Branching rules in the ring of superclass functions of unipotent upper-triangular matrices, J. Algebraic Combin., 31(2):267-298, 2010.

7 Nathaniel Thiem and Vidya Venkateswaran, Restricting supercharacters of the finite group of unipotent uppertriangular matrices. Electron. J. Combin., 16(1):Research Paper 23, 32, 2009.

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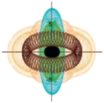
Gauss's Hidden Menagerie:
From Cyclotomy to Supercharacters

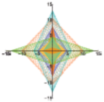
Stephan Ramon Garcia, Trevor Hyde, and Bob Lutz

At the age of eighteen, Gauss established the constructibility of the 17-gon, a result that had eluded mathematicians for two millennia. At the heart of his argument was a keen study of certain sums of complex exponentials, known now as *Gaussian periods*. These sums play starring roles in applications both classical and modern, including Kummer's development of arithmetic in the cyclotomic integers [28] and the optimized AKS primality test of H. W. Lenstra and C. Pomerance [1, 32]. In a poetic twist, this recent application of Gaussian periods realizes "Gauss's dream" of an efficient algorithm for distinguishing prime numbers from composites [24].

We seek here to study Gaussian periods from a graphical perspective. It turns out that these classical objects, when viewed appropriately, exhibit a dazzling and eclectic host of visual qualities. Some images contain discretized versions of familiar shapes, while others resemble natural phenomena. Many can be colorized to isolate certain features; for details, see "Cyclic Supercharacters."



(A) $n = 29 \cdot 109 \cdot 113$, $\omega = 8862$, $c = 113$



(B) $n = 37 \cdot 97 \cdot 113$, $\omega = 5507$, $c = 113$

Figure 1. Eye and jewel—images of *cyclic supercharacters* correspond to sets of Gaussian periods. For notation and terminology, see "Cyclic Supercharacters."

Historical Context

The problem of constructing a regular polygon with compass and straight-edge dates back to ancient times. Descartes and others knew that with only these tools on hand, the motivated geometer could draw, in principle, any segment whose length could be written as a finite composition of sums, products, and square roots of rational numbers [18]. Gauss's construction of the 17-gon relied on showing that

$$16\cos\left(\frac{2\pi}{17}\right) = -1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

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Connection with Number Theory

At the age of eighteen, Gauss established the constructibility of the 17-gon, a result that had eluded mathematicians for two millennia. At the heart of his argument was a keen study of certain sums of complex exponentials, known now as **Gaussian periods**. These sums play starring roles in applications both classical and modern, including Kummer's development of arithmetic in the cyclotomic integers [1] and the optimized AKS primality test of H. W. Lenstra and C. Pomerance [2, 3]. In a poetic twist, this recent application of Gaussian periods realizes "Gauss's dream" of an efficient algorithm for distinguishing prime numbers from composites [4].

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2 Manindra Agrawal, Neeraj Kayal, and Nitin Saxena, PRIMES is in p, Annals of Mathematics, 160(2):781-793, 2004.

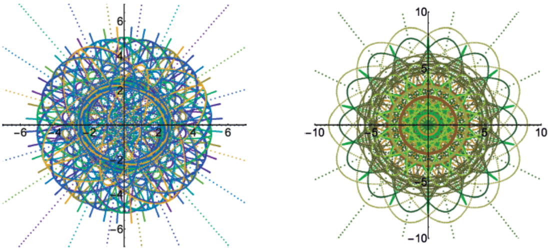
3 H. W. Lenstra, Primality testing with Gaussian periods, Manindra Agrawal and Anil Seth, editors, FST TCS 2002: Foundations of Software Technology and Theoretical Computer Science, volume 2556 of Lecture Notes in Computer Science, Springer, Berlin-Heidelberg, 2002, p. 1.

4 Andrew Granville, It is easy to determine whether a given integer is prime, Bull. Amer. Math. Soc., 42(1):3- 38, 2005.

Cyclic Supercharacter Theory

$$\sigma_X : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$$

$$\sigma_X(y) = \sum_{x \in X} e\left(\frac{xy}{n}\right).$$



(A) $n = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, $\omega = 254$, $c = 11$

(B) $n = 3^2 \cdot 5^2 \cdot 7 \cdot 17^2$, $\omega = 3599$, $c = 17^2$

Figure 4. Atoms—images of *cyclic supercharacters* correspond to sets of Gaussian periods. For notation and terminology, see "Cyclic Supercharacters."

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P. Diaconis and I. M. Isaacs (2008)

Let G be a finite group, and suppose that $\mathcal{K}$ is a partition of G into unions of conjugacy classes and $\mathcal{X}$ is a set of characters of G. We say that the pair $(\mathcal{K}; \mathcal{X})$ is a supercharacter theory of G if

1. $|\mathcal{X}| = |\mathcal{K}|$,
2. the characters $\chi \in \mathcal{X}$ are constant on the members of $\mathcal{K}$, and

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3. each irreducible character of G is a constituent of exactly one character in $\mathcal{X}$.

The characters $\chi \in \mathcal{X}$ are referred to as supercharacters and the sets $K \in \mathcal{K}$ are called superclasses.

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Lemma

Let $(\mathcal{K};\mathcal{L})$ be a supercharacter theory of G. Then

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$$\chi = a_\chi \sum_{\varphi \in S_\chi} \varphi(1)\varphi$$

for some constant a_χ *and some subset* $S_\chi \subset Irr(G)$*.*

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Some supercharacter theories of G:

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Table: The Character Table of A_6.

Class	e^G	x_2^G	x_3^G	x_4^G	x_5^G	x_6^G	x_7^G
Size	1	45	90	40	40	72	72
Order	1	2	4	3	3	5	5
1_G	1	1	1	1	1	1	1
χ_2	5	1	-1	-1	2	0	0
χ_3	5	1	-1	2	-1	0	0
χ_4	10	-2	0	1	1	0	0
χ_5	8	0	0	-1	-1	$1+\alpha$	$-\alpha$
χ_6	8	0	0	-1	-1	$-\alpha$	$1+\alpha$
χ_7	9	1	1	0	0	-1	-1

($\alpha = (\zeta_5)^3 + (\zeta_5)^2$, where ζ_5 is a primitive 5^{th} root of 1)

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Size	1	45	90	40	40	72	72
Order	1	2	4	3	3	5	5
1_G	1	1	1	1	1	1	1
$\sum_{i=2}^{4} \chi_i(1)\chi_i$	150	-10	-10	15	15	0	0
$\sum_{i=5}^{6} \chi_i(1)\chi_i$	128	0	0	-16	-16	8	8
$\chi_7(1)\chi_7$	81	9	9	0	0	-9	-9

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	e^G	$x_2^G \cup x_3^G$	$x_4^G \cup x_5^G$	$x_6^G \cup x_7^G$
1_G	1	1	1	1
$\sigma_X = \sum_{i=2}^{4} \chi_i(1)\chi_i$	150	-10	15	0
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1_G	1	1	1	1
$\chi_X = \chi_2 + \chi_3 + 2\chi_4$	30	-2	3	0
$\chi_Y = \chi_5 + \chi_6$	16	0	-2	1
χ_7	9	1	0	-1

Any group $H \leq Aut(G)$:
Take $\mathcal{K}$ to be the orbits on classes of G
and $\mathcal{X}$ to be the orbits on $Irr(G)$.
(Then $|\mathcal{K}| = |\mathcal{X}|$ by a lemma of Brauer.)

Lemma

Suppose G is a finite group, $A = \{\chi(x) \mid \chi \in Irr(G)\}$ and $\mathbb{Q}(A)$ denotes the field generated by $\mathbb{Q}$ and A. Then the following holds:

1. *If $\mathcal{X}(G) = \{\{\chi, \overline{\chi}\} \mid \chi \in Irr(G)\}$ and $\mathcal{K}(G) = \{\{x^G, (x^{-1})^G\} \mid x \in G\}$ then $(\mathcal{X}(G), \mathcal{K}(G))$ is a supercharacter theory of G.*
2. *If $\Gamma = Gal(\frac{\mathbb{Q}(\mathbb{A})}{\mathbb{Q}})$, $\mathcal{X}(G)$ is the set of all orbits of Γ on $Irr(G)$ and $\mathcal{K}(G)$ is the set of all orbits of Γ on $Con(G)$ then $(\mathcal{X}(G), \mathcal{K}(G))$ is a supercharacter theory of G.*

A. R. Ashrafi, F. Koorepazan-Moftakhar, Towards the classification of finite simple groups with exactly three or four supercharacter theories, Asian-Eur. J. Math. 11(5):1850096 (2018).

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Let $X = \{\chi_1, \ldots, \chi_u\}$ and $K = \{K_1, \ldots, K_s\}$ be parts of set partitions $\mathcal{X}$ of $Irr(G)$ and $\mathcal{K}$ of $Con(G)$, respectively. If $\sigma_X(K_1) = \cdots = \sigma_X(K_s)$, then we say that X and K are *consistent*. If all parts of $\mathcal{X}$ are mutually consistent with all parts of $\mathcal{K}$, then the set partitions $\mathcal{X}$ and $\mathcal{K}$ are said to be consistent.

A part X of a set partition $\mathcal{X}$ of $Irr(G)$ is said to be *bad* if X is consistent with only singleton subsets of $Con(G)$. A set partition containing a bad part is called a *bad set partition*.
It is easy to see that a bad set partition $\mathcal{X}$ of $Irr(G)$ does not have a mate $\mathcal{K}$ such that $(\mathcal{X}, \mathcal{K}) \in Sup(G)$.

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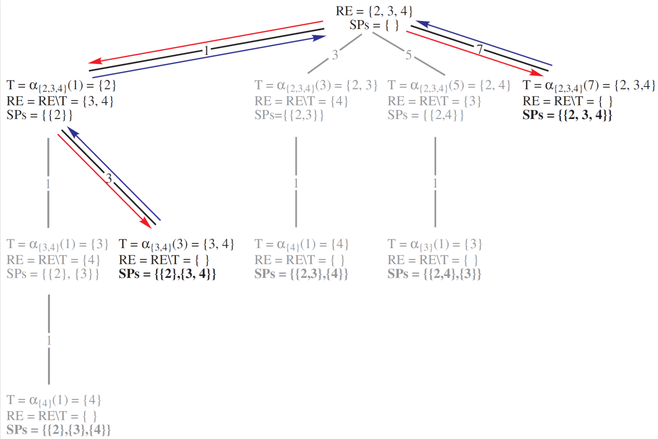
Tabelle: Percentage of existence of bad parts for some cyclic and dihedral groups.

$\kappa(G)$	Group	$\alpha(G)$	$\|Sup(G)\|$	$\kappa(G)$	Group	$\alpha(G)$	$\|Sup(G)\|$
4	D_4	%0	5	2	Z_2	%100	1
3	D_6	%66.67	2	3	Z_3	%66.67	2
4	D_{10}	%57.14	3	5	Z_5	%80	3
5	D_{14}	%80	3	7	Z_7	%85.7	4
7	D_{22}	%95	3	11	Z_{11}	%96.77	4
8	D_{26}	%84.3	5	13	Z_{13}	%98.16	6
10	D_{34}	%93.75	5	17	Z_{17}	%99.6	5
11	D_{38}	%98.53	4	19	Z_{19}	%99.78	6
13	D_{46}	%99.92	3				
16	D_{58}	%99.2	5				
17	D_{62}	%99.88	5				

History of algorithms for computing set partitions In literature, there are two algorithms by Semba and Er for computing set partitions of $[n]$.

Tabelle: Order of algorithms for computing set partitions

Semba (1984)	Er (1988)	(2008)
$\Theta(4B_n)$	$\Theta(1.6B_n)$	$\frac{\sum_{i=2}^{n}}{B_n} < 2$



CreateSetPartitions([4]*, badparts=[[2,3]; [2,4]; [3]; [4]])

The average running time after three runs for both algorithms on a computer with

1. Processor: Intel® Core(TM) m7-6Y75 CPU @ 1.20 GHz 1.51 GHz,
2. installed memory (RAM) 8.00 GB (7.90 GB usable),
3. with system type: 64 bit operating system, x64-based processor

are summarized in the next table:

Tabelle: Comparing the running times for some groups.

$\kappa(G)$	G	$\lvert Sup(G)\rvert$	$\lvert BP(G)\rvert$	$\alpha(G)$	Main algorithm (MA)(second)	First algorithm (FA)(second)	FA/MA
10	[100, 11]	623	0	0	1.4	1.1	0.8
11	[32, 43]	376	0	0	7.6	6.7	0.9
11	[32, 44]	376	0	0	8.1	7.5	0.9
12	[1296, 3523]	1058	0	0	70.1	62	0.9
13	[64, 32]	325	0	0	464.6	429.8	0.9
12	D36	51	168	8.2	65.4	68.2	1.04
12	M22	5	288	14.1	65.8	61.8	0.9
10	M11	5	112	21.9	0.8	1.1	1.4
10	D28	23	144	28.9	0.8	1.7	2.1
10	[120, 35]	10	152	29.7	0.6	1.1	1.8
13	[93, 1]	9	1980	48.4	169.7	662.3	3.9
13	[253, 1]	9	1980	48.4	127.8	521.3	4.08
10	Z10	10	376	73.6	0.06	1.2	20
10	D34	5	480	93.9	0.04	1.3	32.5
11	Z11	4	990	96.8	0.1	7.9	79
13	Z13	6	4020	98.1	1.2	548.2	456.8
11	D38	4	1008	98.5	0.1	8.5	85
13	D46	3	4092	99.9	1.2	610.6	508.8

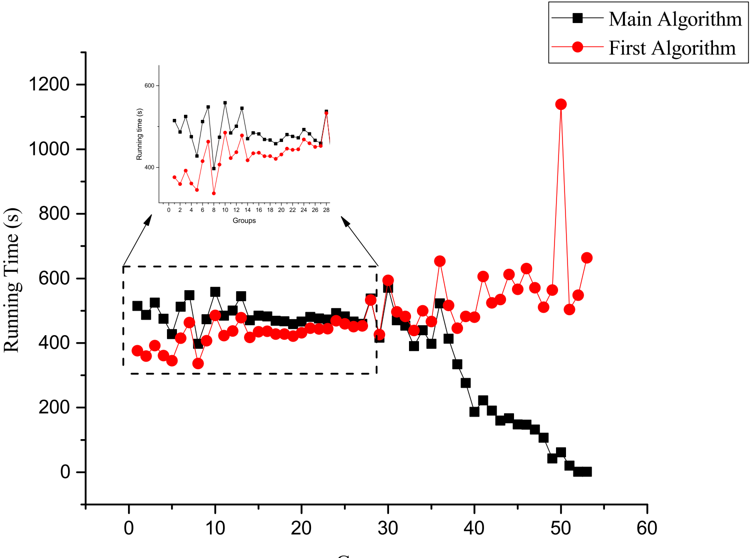

Main Algorithm
First Algorithm
Running Time (s)
1200
1000
800
600
400
200
0
0
10
20
30
40
50
60
Running time (s)
600
400
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28
Groups

Tabelle: The GAP id of all groups with exactly 13 conjugacy classes.

1	[162, 21]	12	[162, 20]	23	[960, 11359]	34	[100, 10]	45	[328, 12]
2	[96, 191]	13	[1944, 2290]	24	[64, 33]	35	[162, 15]	46	[148, 3]
3	[400, 206]	14	[64, 37]	25	[1000, 86]	36	[40, 6]	47	[333, 3]
4	[162, 22]	15	[64, 32]	26	[720, 409]	37	[1053, 51]	48	[156, 7]
5	[192, 1494]	16	[96, 190]	27	[576, 8652]	38	[120, 38]	49	[301, 1]
6	[96, 193]	17	[192, 1491]	28	[40, 4]	39	[600, 148]	50	[205, 1]
7	[1944, 2289]	18	[64, 36]	29	**[162,11]**	40	[216, 86]	51	[150, 5]
8	[192, 1493]	19	[192, 1492]	30	[160, 199]	41	[258, 1]	52	[13, 1]
9	[1440, 5841]	20	[64, 35]	31	[324, 160]	42	[310, 1]	53	[46, 1]
10	[40, 8]	21	[162, 19]	32	[162, 13]	43	[253, 1]		
11	[64, 34]	22	[216, 87]	33	[1320, 133]	44	[93, 1]		

1. NrSuperCharacterTheories(G);
2. SuperCharacterTheories(G);

1. NrSuperCharacterTheories(G);
2. SuperCharacterTheories(G);
3. SuperCharacterTable(G);

1. NrSuperCharacterTheories(G);
2. SuperCharacterTheories(G);
3. SuperCharacterTable(G);

Thanks for your attention

Tabatabaei Historical House, Kashan, Iran