Finite local nearrings and their classification

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Nearrings are generalization of associative rings, in which the additive group can be non-abelian, and addition is connected with multiplication by only one distributive law, left or right. In this sense local nearrings are generalization of local rings.

Definition

A set R with two binary operations "$+$" and "$\cdot$" is called a (left) nearring if the following statements hold:

(1) $(R,+)$ is a (not necessarily abelian) group with neutral element 0;

(2) $(R,\cdot)$ is a semigroup;

(3) $x\cdot(y+z) = x\cdot y + x\cdot z$ for all x, y, $z \in R$.

Such a nearring is called a left nearring. If axiom 3) is replaced by an axiom $(x+y)\cdot z = x\cdot z + y\cdot z$ for all x, y, $z \in R$, then we get a right nearring.

Clearly every associative ring is a nearring and each group is the additive group of a nearring, but not necessarily of a nearring with identity. The question what group can be the additive group of a nearring with identity is far from solution.

Boykett and Nöbauer [1] classyfied all non-abelian groups of order less than 32 that can be the additive groups of a nearring with identity and found the number of non-isomorphic nearrings with identity on such groups (see also GAP [2] package SONATA [3]).

GAP is an open source project, started in RWTH Aachen in late 80s. Since 1997 its development has been coordinated by CIRCA (the Centre for Interdisciplinary Research in Computational Algebra) at the University of St Andrews. GAP provides a programming language, also called GAP, a library of mathematical algorithms implemented in this language, and various libraries of mathematical objects, such as, for example, the 423164062 groups of order not greater than 2000 (excluding 49487365422 groups of order 1024).

Definition 2.

A nearring R is called local, if the set L of all non-invertible elements of R forms a subgroup of its additive group R^+.

A study of local nearrings was initiated by Maxson [4] who defined a number of their basic properties and proved in particular that the additive group of a finite zero-symmetric local nearring is a p-group.

The package SONATA of the computer algebra system GAP contains a library of all non-isomorphic nearrings of order at most 15 and nearrings with a unity of order up to 31, among which 698 are local. However, the classification of nearrings of higher orders requires much more complex calculations. For local nearrings they were realized in the form of a new GAP package called LocalNR [5]. Its current version (not yet distributed with GAP) contains 37599 local nearrings of order at most 361, except orders 128, 256 and some of orders 32, 64 and 243. We have already calculated some classes of local nearrings of orders 32 (with 14927685 nearrings), 64 (with 1115947 nearrings) and 243 (with 705105 nearrings).

Let $[n,i]$ be the i-th group of order n in the SmallGroups library in the computer system algebra GAP. We denote by C_n the cyclic group of order n.

There are 51 non-isomorphic groups of order 32 from which 19 are the additive groups.

Theorem 1 (Raievska I., Raievska M., Sysak Ya. [6])

The following groups and only they are the additive groups of local nearrings of order 32:

IdGroup	Structure Description	Number of LNR
[32,1]	C_{32}	1
[32,2]	$(C_4 \times C_2) \rtimes C_4$	1397
[32,3]	$C_8 \times C_4$	880
[32,4]	$C_8 \rtimes C_4$	798
[32,5]	$(C_8 \times C_2) \rtimes C_2$	1945
[32,6]	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2$	433
[32,7]	$(C_8 \rtimes C_2) \rtimes C_2$	225
[32,8]	$C_2.((C_4 \times C_2) \rtimes C_2)$	208

[32,12]	$C_4 \rtimes C_8$	2406
[32,16]	$C_{16} \times C_2$	129
[32,17]	$C_{16} \rtimes C_2$	129
[32,21]	$C_4 \times C_4 \times C_2$	135558
[32,22]	$C_2 \times ((C_4 \times C_2) \rtimes C_2)$	149374
[32,23]	$C_2 \times (C_4 \rtimes C_4)$	157905
[32,24]	$(C_4 \times C_4) \rtimes C_2$	262544
[32,36]	$C_8 \times C_2 \times C_2$	177175
[32,37]	$C_2 \times (C_8 \rtimes C_2)$	527419
[32,45]	$C_4 \times C_2 \times C_2 \times C_2$	exact number is unknown
[32,51]	$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	exact number is unknown

There exist 267 non-isomorphic groups of order $64 = 2^6$ from which 53 are 2-generated groups and only 24 of these groups are the additive groups of local nearrings.

Theorem 2 (Raievska I., Raievska M., Sysak Ya. [7])

The following 2-generated groups are the additive groups of local nearrings of order 64:

IdGroup	Structure Description	Number of LNR
$[64, 2]$	$C_8 \times C_8$	1683
$[64, 3]$	$C_8 \rtimes C_8$	1628
$[64, 4]$	$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	167245
$[64, 5]$	$(C_4 \times C_2) \rtimes C_8$	118190
$[64, 10]$	$(C_8 \rtimes C_4) \rtimes C_2$	7424
$[64, 12]$	$(C_4 \rtimes C_8) \rtimes C_2$	5633
$[64, 14]$	$(C_2 \times C_2).((C_4 \times C_2) \rtimes C_2)$	5520
$[64, 15]$	$C_8 \rtimes C_8$	2384
$[64, 16]$	$C_8 \rtimes C_8$	2384
$[64, 17]$	$(C_8 \times C_2) \rtimes C_4$	433060
$[64, 23]$	$(C_4 \times C_2 \times C_2) \rtimes C_4$	111758

$[64, 24]$	$(C_8 \rtimes C_2) \rtimes C_4$	109189
$[64, 26]$	$C_{16} \times C_4$	11467
$[64, 27]$	$C_{16} \rtimes C_4$	11467
$[64, 29]$	$(C_{16} \times C_2) \rtimes C_2$	28185
$[64, 30]$	$(C_{16} \rtimes C_2) \rtimes C_2$	4433
$[64, 34]$	$(((C_4 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	16177
$[64, 35]$	$(C_4 \times C_4) \rtimes C_4$	15504
$[64, 36]$	$(C_2.((C_4 \times C_2) \rtimes C_2)$	15761
$[64, 37]$	$C_2.(((C_4 \times C_2) \rtimes C_2) \rtimes C_2)$	15920
$[64, 44]$	$C_4 \rtimes C_{16}$	28500
$[64, 45]$	$C_8.D_8 = C_4.(C_8 \times C_2)$	1920
$[64, 50]$	$C_{32} \times C_2$	257
$[64, 51]$	$C_{32} \rtimes C_2$	257

There exist 2328 non-isomorphic groups of order $128 = 2^7$ from which 162 are 2-generated groups (5 groups are of exponent 64, 18 groups are of exponent 32, 65 groups are of exponent 16, 72 groups are of exponent 8, and 2 groups are of exponent 4).

Theorem 3 (Raievska I., Raievska M., 2022)

The following 2-generated groups of exponent 4 and only they are the additive groups of zero-symetric local nearrings of order 128:

IdGroup	Structure Description	Number of LNR
[128, 36]	$(C_2 \times ((C_4 \times C_2) \rtimes C_2)) \rtimes C_4$	> 80384
[128, 125]	$(C_4 \times C_4 \times C_2) \rtimes C_4$	> 35040

Proposition 1.

The following 2-generated groups of exponent 8 are the additive groups of zero-symmetric local nearrings of order 128:

IdGroup	Structure Description	Number of LNR
$[128,2]$	$((C_8 \times C_2) \rtimes C_4) \rtimes C_2$	> 41184
$[128,4]$	$(C_2 \times Q_8) \rtimes C_8$	> 46912
$[128,5]$	$C_8 \times C_2) \rtimes C_8$	> 1536
$[128,6]$	$(C_8 \times C_4) \rtimes C_4$	> 73728
$[128,7]$	$(C_8 \times C_2) \rtimes C_8$	> 4160
$[128,8]$	$(C_4 \rtimes C_8) \rtimes C_4$	> 10240
$[128,12]$	$((C_8 \times C_2) \rtimes C_2) \rtimes C_4$	> 1336
$[128,13]$	$(C_8 \times C_2) \rtimes C_8$	> 19136
$[128,27]$	$(C_8 \rtimes C_4) \rtimes C_4$	> 20736
$[128,38]$	$((C_8 \times C_2) \rtimes C_2) \rtimes C_4$	> 80384
$[128,48]$	$(((C_8 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	> 102240
$[128,49]$	$(C_4 \times C_2 \times C_2) \rtimes C_8$	> 99680
$[128,50]$	$((C_4 \times C_2) \rtimes C_8) \rtimes C_2$	> 16992
$[128,51]$	$(C_2 \times Q_8) \rtimes C_8$	> 16992
$[128,56]$	$(C_4 \times C_4) \rtimes C_8$	> 196608
$[128,57]$	$(C_4 \times C_4) \rtimes C_8$	> 127488

Question 1.

Are the following 2-generated groups of exponent 8 the additive groups of zero-symmetric local nearrings of order 128?

IdGroup	Structure Description
$[128, 9]$	$(C_8 \times C_2) \rtimes C_8$
$[128, 28]$	$(C_4 \rtimes C_8) \rtimes C_4$
$[128, 126]$	$(C_2.((C_4 \times C_2) \rtimes C_2) = (C_2 \times C_2).(C_4 \times C_2)) \rtimes C_4$

Theorem 4 (Raievska I., Raievska M., 2022)

The following 2-generated groups of exponent 16 and only they are the additive groups of zero-symetric local nearrings of order 128:

IdGroup	Structure Description	Number of LNR
[128, 42]	$C_{16} \times C_8$	>134754
[128, 43]	$C_{16} \rtimes C_8$	>133866
[128, 44]	$C_8 \rtimes C_{16}$	>145648
[128, 46]	$((C_{16} \times C_2) \rtimes C_2) \rtimes C_2$	>24704
[128, 47]	$((C_{16} \times C_2) \rtimes C_2) \rtimes C_2$	252928
[128, 52]	$((C_{16} \rtimes C_2) \rtimes C_2) \rtimes C_2$	>115840

[128, 53]	$((C_{16} \rtimes C_2) \rtimes C_2) \rtimes C_2$	>277248
[128, 54]	$(C_4 \times C_2) \rtimes C_{16}$	>82944
[128, 55]	$(C_4 \times C_2).((C_4 \times C_2) \rtimes C_2) = (C_4 \times C_2).(C_8 \times C_2)$	640
[128, 59]	$C_4.((C_2 \times C_2 \times C_2) \rtimes C_4) = (C_4 \times C_2).(C_8 \times C_2)$	>13056
[128, 99]	$C_8 \rtimes C_{16}$	>29248
[128, 102]	$C_8 \rtimes C_{16}$	>5376
[128, 106]	$(C_{16} \times C_2) \rtimes C_4$	>2808
[128, 107]	$(C_{16} \times C_2) \rtimes C_4$	>16460
[128, 108]	$(C_{16} \rtimes C_2) \rtimes C_4$	>1344
[128, 109]	$(C_{16} \rtimes C_2) \rtimes C_4$	>2344

Theorem 5 (Raievska I., Raievska M., 2022)

There exist 389976 zero-symetric local nearrings on 2-generated additive groups of exponent 32 of order 128:

IdGroup	Structure Description	Number of LNR
[128, 128]	$C_{32} \times C_4$	48968
[128, 129]	$C_{32} \rtimes C_4$	48968
[128, 131]	$(C_{32} \times C_2) \rtimes C_2$	144016
[128, 132]	$(C_{32} \rtimes C_2) \rtimes C_2$	23936
[128, 153]	$C_4 \rtimes C_{32}$	118968
[128, 154]	$C_{16}.D_8 = C_4.(C_{16} \times C_2)$	5120

Theorem 6 (Raievska I., Raievska M., 2022)

There exist 1024 zero-symetric local nearrings on 2-generated additive groups of exponent 64 of order 128:

IdGroup	Structure Description	Number of LNR
[128, 159]	$C_{64} \times C_2$	512
[128, 160]	$C_{64} \rtimes C_2$	512

The library of zero-symmetric local nearrings of order 128 on 2-generated groups can be extracted from [8] using the package LocalNR.

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