Constructive Category Theory and Applications in Algebraic Geometry

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Outline

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Constructive category theory

Abstraction of language

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Addition of two numbers:

Data type: `int`

Data type: `float`

Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

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Abstraction of language

Addition of two numbers: Assembly

Data type: `int`

```
addi:
movl   %edi, -4(%rsp)
movl   %esi, -8(%rsp)
movl   -4(%rsp), %esi
addl   -8(%rsp), %esi
movl   %esi, %eax
ret
```

Data type: `float`

Abstraction of language

Addition of two numbers: Assembly

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addi:
movl  %edi, -4(%rsp)
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movl  -4(%rsp), %esi
addl  -8(%rsp), %esi
movl  %esi, %eax
ret
```

Data type: `float`

```
addf:
movss %xmm0, -4(%rsp)
movss %xmm1, -8(%rsp)
movss -4(%rsp), %xmm0
addss -8(%rsp), %xmm0
ret
```

Abstraction of language

Addition of two numbers: C

Data type: `int`

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Abstraction of language

Addition of two numbers: C

Data type: `int`

```
int addi( int a,
          int b )
{
  return a + b;
}
```

Data type: `float`

Abstraction of language

Addition of two numbers: C

Data type: `int`

```
int addi( int a,
          int b )
{
  return a + b;
}
```

Data type: `float`

```
float addf( float a,
            float b )
{
  return a + b;
}
```

Abstraction of language

Addition of two numbers: GAP or Julia

Data type: `int`

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Abstraction of language

Addition of two numbers: GAP or Julia

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```
function( a, b )
    return a + b;
end;
```

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High language leads to generic code!

Abstraction of language

Computing the intersection of two subobjects

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Solution of

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Euclidean algorithm:

$$\langle \mathrm{lcm}\,(x, y) \rangle$$

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Generic algorithm for both cases?

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Generic algorithm for both cases? **Category theory!**

Category theory as programming language

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CAP - Categories, Algorithms, and Programming

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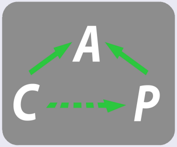

CAP implements a
categorical programming language

Categories

Definition

A category $\mathcal{A}$ contains the following data:

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$A \qquad B \qquad C$

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Computable categories

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1 2 1

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$1 \qquad 2 \qquad 1$

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$$1 \xrightarrow[\begin{pmatrix} 1 & 2 \end{pmatrix}]{} 2 \xrightarrow[\begin{pmatrix} 3 \\ 4 \end{pmatrix}]{} 1$$

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$$\begin{tikzcd} 1 \arrow[r, "\begin{pmatrix} 1 & 2 \end{pmatrix}"'] \arrow[rr, bend left, "\begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = (11)"] & 2 \arrow[r, "\begin{pmatrix} 3 \\ 4 \end{pmatrix}"'] & 1 \end{tikzcd}$$

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Categorical operations

Some categorical operations in abelian categories

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Implementation of the kernel

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Let $\varphi \in \mathrm{Hom}(A, B)$. To fully describe the kernel of φ ...

... one needs an object $\ker \varphi$,

$$\begin{array}{ccc} \ker \varphi & & \\ & A \xrightarrow{\varphi} B \end{array}$$

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Let $\varphi \in \mathrm{Hom}(A, B)$. To fully describe the kernel of φ ...

... one needs an object $\ker \varphi$,
its embedding $\kappa = \mathrm{KernelEmbedding}(\varphi)$,

$$\begin{tikzcd} \ker \varphi \arrow[dr, hook, "\kappa"] & & \\ & A \arrow[r, "\varphi"] & B \end{tikzcd}$$

Implementation of the kernel

Let $\varphi \in \mathrm{Hom}(A, B)$. To fully describe the kernel of φ ...

... one needs an object $\ker\varphi$,
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and for every test morphism τ

$$\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] \arrow[drr, bend left, "0"] & & \\
& A \arrow[r, "\varphi"] & B \\
T \arrow[ur, "\tau"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

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a *unique* morphism $\lambda = \mathrm{KernelLift}(\varphi, \tau)$

$$
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Implementation of the kernel: $\mathbb{Q}$-vector spaces

$\mathrm{Obj} := \mathbb{Z}_{\geq 0}$, $\mathrm{Hom}\,(m, n) := \mathbb{Q}^{m \times n}$

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Compute

- $\ker \varphi$ as $\dim(A) - \mathrm{rank}(\varphi)$

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CAP is a framework to implement computable categories and provides

- specifications of categorical operations,
- generic algorithms based on basic categorical operations,
- a categorical programming language having categorical operations as syntax elements.

Computing the intersection

Let $M_1 \subseteq N$ and $M_2 \subseteq N$ subobjects.

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$$\begin{tikzcd}
M_1 \arrow[dr, hook, "\iota_1"] & \\
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- $\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2$

Computing the intersection

Let $M_1 \hookrightarrow N$ and $M_2 \hookrightarrow N$ subobjects.
Compute their intersection $\gamma : M_1 \cap M_2 \hookrightarrow N$.

$$\begin{tikzcd}
 & & M_1 \arrow[dr, hook, "\iota_1"] & \\
M_1 \cap M_2 \arrow[r, hook, "\kappa"] & M_1 \oplus M_2 \arrow[ur, two heads, "\pi_1"] \arrow[dr, two heads, "\pi_2"'] \arrow[rr, "\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2"] & & N \\
 & & M_2 \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}$$

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$$\pi_i := \mathrm{ProjectionInFactorOfDirectSum}\left(\left(M_1, M_2\right), i\right),\ i = 1, 2$$

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```
pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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```

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```

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```
lambda := PostCompose( iota1, pi1 );
phi := lambda - PostCompose( iota2, pi2 );
```

$\kappa := \text{KernelEmbedding}\left(\varphi\right)$

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```
kappa := KernelEmbedding( phi );
```

$\gamma := \iota_1 \circ \pi_1 \circ \kappa$

```
gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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Translation to CAP

```
Schnitt := function( iota1, iota2 )



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  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
Schnitt := function( iota1, iota2 )

  M1 := Source( iota1 );
  M2 := Source( iota2 );

  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );

  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );

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  gamma := PostCompose( lambda, kappa );
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Translation to CAP

```
Schnitt := function( iota1, iota2 )

  M1 := Source( iota1 );
  M2 := Source( iota2 );
  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
  return gamma;
end;
```

Translation to CAP

```
Schnitt := function( iota1, iota2 )
  local M1, M2, pi1, pi2, lambda, phi, kappa, gamma;
  M1 := Source( iota1 );
  M2 := Source( iota2 );
  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
  return gamma;
end;
```

Computing the intersection: $\mathbb{Q}$-vector space

Compute the intersection of

$$\begin{array}{ccccc} M_1 & \xhookrightarrow{\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}} & N & \xhookleftarrow{\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} & M_2 \\ \shortparallel & & \shortparallel & & \shortparallel \\ 2 & & 3 & & 2 \end{array}$$

Computing the intersection: $\mathbb{Q}$-vector space

Compute the intersection of

$$
\begin{tikzcd}
M_1 \arrow[r, hook, "\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \arrow[d, phantom, "\shortparallel"] & N \arrow[d, phantom, "\shortparallel"] & M_2 \arrow[l, hook', "\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}"'] \arrow[d, phantom, "\shortparallel"] \\
2 & 3 & 2
\end{tikzcd}
$$

```
gap> gamma := Schnitt( iota1, iota2 );
<A morphism in the category of matrices over Q>
```

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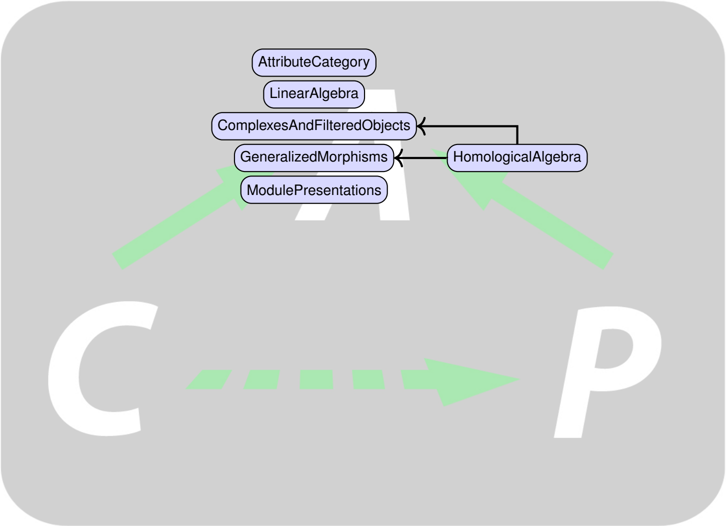
$$\begin{tikzcd} M_1 \arrow[r, hook, "\iota_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \arrow[d, phantom, "\shortparallel"] & N \arrow[d, phantom, "\shortparallel"] & M_2 \arrow[l, hook', "\iota_2 := \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}"'] \arrow[d, phantom, "\shortparallel"] \\ 2 & 3 & 2 \end{tikzcd}$$

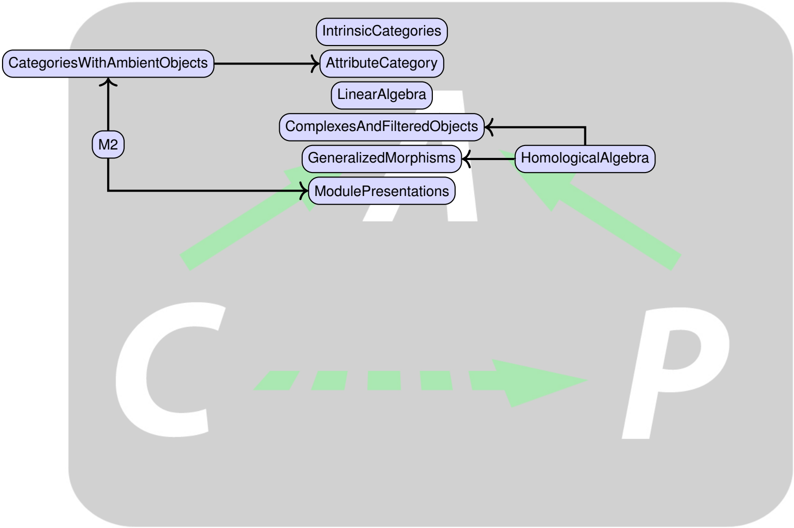
```
gap> gamma := Schnitt( iota1, iota2 );
<A morphism in the category of matrices over Q>
gap> Display( gamma );
[ [  1,  1,  0 ] ]

A morphism in the category of matrices over Q
```

CAP packages

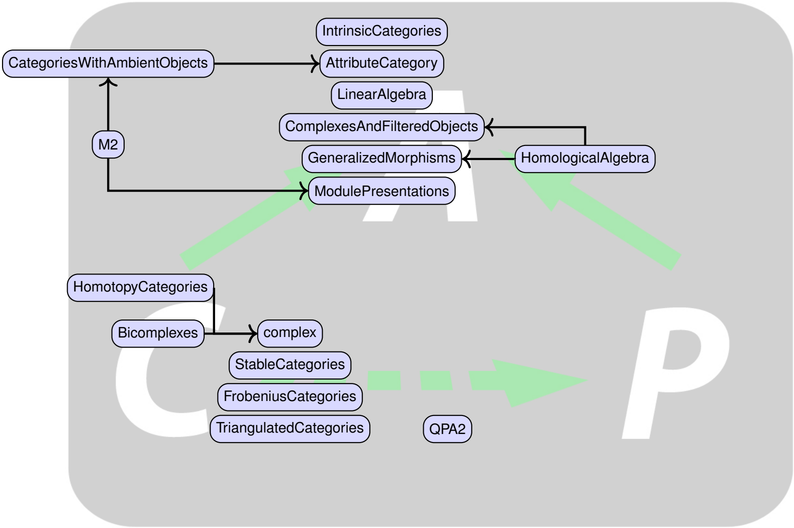
CAP packages



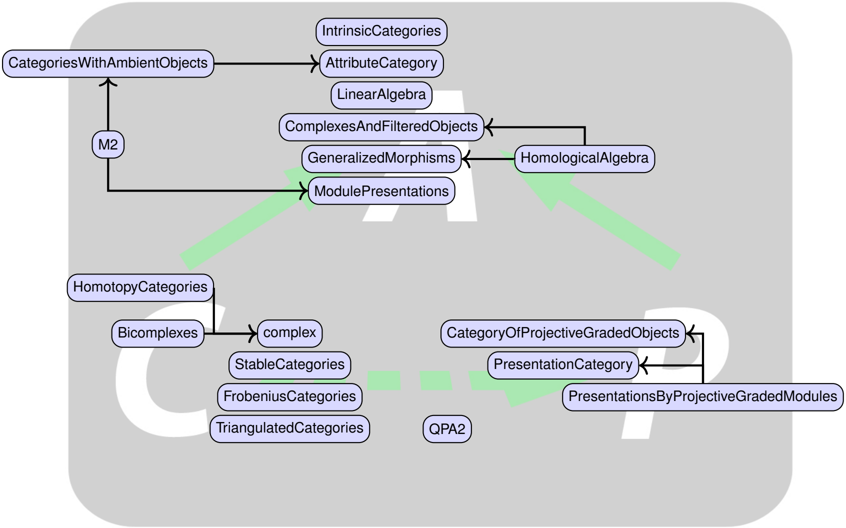
CAP packages



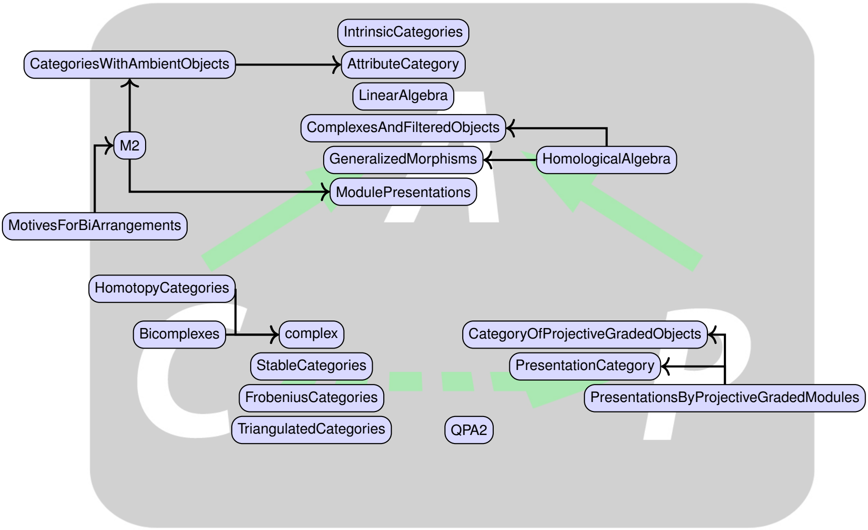
CAP packages



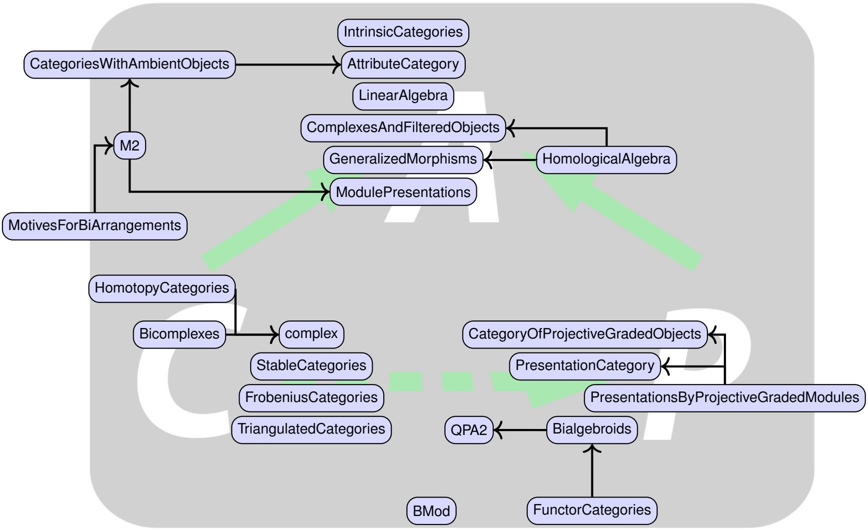
CAP packages



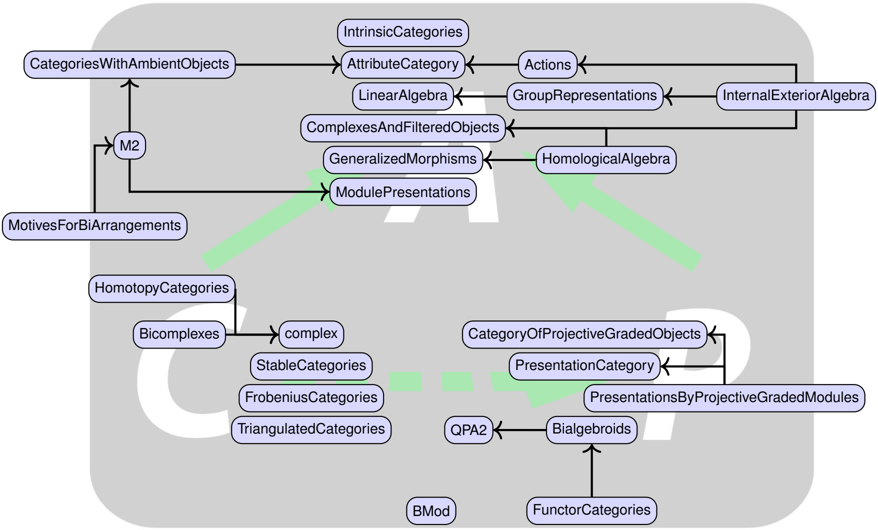
CAP packages



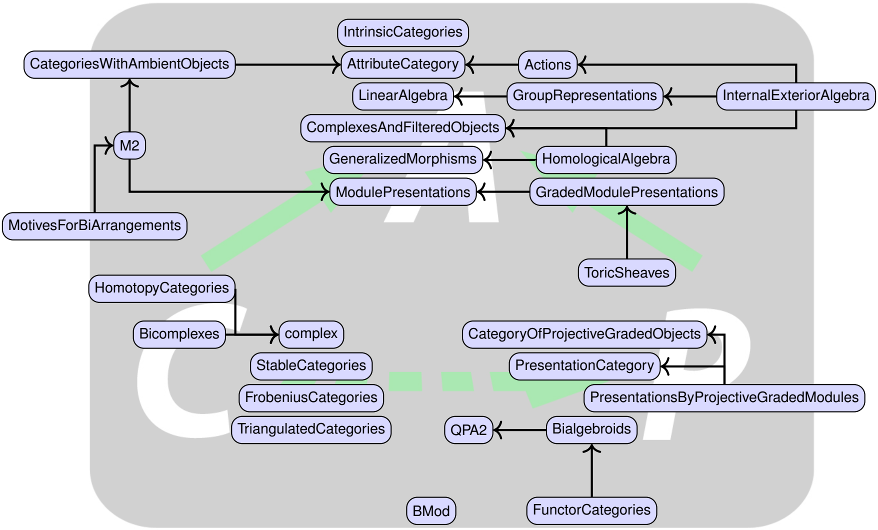
CAP packages



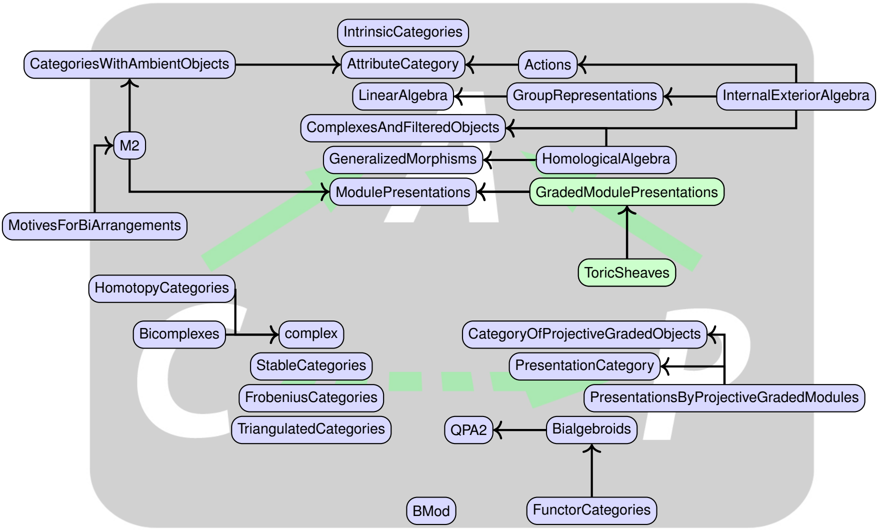
CAP packages



CAP packages



CAP packages



Applications to Algebraic Geometry

Coherent sheaves: Affine space

Let K be an algebraically closed field.

Coherent sheaves: Affine space

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Affine space

- Affine space: $\mathbb{A}^n = K^n$

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In the language of category theory:

Coherent sheaves: Affine space

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In the language of category theory:
Equivalence of categories

$$S\text{-mod} \xrightarrow{\sim} \mathfrak{Coh}(\mathbb{A}^n)$$

Coherent sheaves

Coherent sheaves

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Coherent sheaves

Normal toric variety (smooth)

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Coherent sheaves

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Coherent sheaves

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Equivalence of categories

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In the language of category theory:
Equivalence of categories

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Coherent sheaves

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- Toric variety $X = (K^n/G') - Z$, $G' \cong \mathrm{Hom}\,(G, K^*)$
- Coherent sheaves correspond to f. g. modules over $S := K[x_1, \dots, x_n]$ with a G-grading modulo modules that are only supported on Z.

In the language of category theory:
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$$S\text{-grmod}_G / S\text{-grmod}_G^0 \xrightarrow{\sim} \mathfrak{Coh}\,(X)$$

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Computability of $S\text{-grmod}_G / S\text{-grmod}_G^0$?

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`FiberProduct`: Algorithm for intersection

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Composition only by computations in $\mathcal{A}$!

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