Computing with Semigroup Congruences

Congruences of finite simple and 0-simple semigroups in GAP

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Semigroups

Definition

A **semigroup** is a set S together with a binary operation $* : S \times S \to S$ such that

$$(x * y) * z \quad = \quad x * (y * z)$$

for all $x, y, z \in S$.

Congruences

Definition

A **congruence** on a semigroup S is a relation $\rho \subseteq S \times S$ such that

(R) $(x, x) \in \rho$,

(S) $(x, y) \in \rho \quad \Rightarrow \quad (y, x) \in \rho$,

(T) $(x, y), (y, z) \in \rho \quad \Rightarrow \quad (x, z) \in \rho$,

(C) $(x, y) \in \rho \quad \Rightarrow \quad (ax, ay), (xa, ya) \in \rho$,

or equivalently,

(C) $(x, y), (s, t) \in \rho \quad \Rightarrow \quad (xs, yt) \in \rho$,

for all $x, y, z, a, s, t \in S$.

(we may write $x \ \rho \ y$ for $(x, y) \in \rho$)

Simple ways to represent congruences

- List of pairs: $\{(x_1, x_3), (x_1, x_9), (x_{42}, x_{11}), \ldots\}$
- Partition: $\{\{x_1, x_3, x_9, x_{14}\}, \{x_2\}, \{x_4, x_5, x_8\}, \ldots\}$
- ID list: $(1, 2, 1, 3, 3, 4, 5, 3, 1, \ldots)$

Ideals

Let S be a semigroup.

Definition

A **(two-sided) ideal** is a non-empty subset $I \subseteq S$ such that

$$si \in I \quad \text{and} \quad is \in I$$

for all $s \in S$ and $i \in I$.

0-simple semigroups

Definition

A semigroup element $0 \in S$ is called **zero** if

$$0x = x0 = 0$$

for all $x \in S$.

Definition

A semigroup S without zero is **simple** if it has no proper ideals.

Definition

A semigroup S with zero is **0-simple** if its only ideals are $\{0\}$ and S.

Rees 0-matrix semigroups

Definition

A **Rees 0-matrix semigroup** $\mathcal{M}^0[T; I, \Lambda; P]$ is the set

$$(I \times T \times \Lambda) \cup \{0\}$$

with multiplication given by

$$(i, a, \lambda) \cdot (j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where

- T is a semigroup,
- I and Λ are index sets,
- P is a $|\Lambda| \times |I|$ matrix with entries $(p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ taken from T^0,
- $0x = x0 = 0$ for all x in the semigroup.

The Rees Theorem

Theorem (Rees)

Every completely 0-simple semigroup is isomorphic to a Rees 0-matrix semigroup

$$\mathcal{M}^0[G; I, \Lambda; P],$$

where G is a group and P is regular. Conversely, every such Rees 0-matrix semigroup is completely 0-simple.

Linked triples

Definition

For a finite 0-simple Rees 0-matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$, a **linked triple** is a triple

$$(N, \mathcal{S}, \mathcal{T})$$

consisting of a normal subgroup $N \trianglelefteq G$, an equivalence relation $\mathcal{S}$ on I and an equivalence relation $\mathcal{T}$ on Λ, such that the following are satisfied:

1. $\mathcal{S}$ only relates columns which have zeroes in the same places,
2. $\mathcal{T}$ only relates rows which have zeroes in the same places,
3. For all $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $p_{\lambda i}, p_{\lambda j}, p_{\mu i}, p_{\mu j} \neq 0$ and either $(i, j) \in \mathcal{S}$ or $(\lambda, \mu) \in \mathcal{T}$, we have that $q_{\lambda\mu ij} \in N$, where

$$q_{\lambda\mu ij} = p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1}.$$

Linked triples

A finite 0-simple semigroup S has a bijection Γ between its linked triples and its *non-universal* congruences:

$$\Gamma : \rho \mapsto (N, \mathcal{S}, \mathcal{T})$$

Two non-zero elements (i, a, λ) and (j, b, μ) are ρ-related if and only if

1. $(i, j) \in \mathcal{S}$;
2. $(\lambda, \mu) \in \mathcal{T}$;
3. $(p_{\xi i} a p_{\lambda x})(p_{\xi j} b p_{\mu x})^{-1} \in N$ for some $x \in I, \xi \in \Lambda$ such that $p_{\xi i}, p_{\xi j}, p_{\lambda x}, p_{\mu x} \neq 0$.

Semigroups package

- Generic semigroups: **generating pairs**
- Simple & 0-simple semigroups: **linked triples**
- Inverse semigroups: **kernel and trace**